Derivations of Applied Mathematics

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Revised 14 December 2006

Derivations of Applied Mathematics.
14 December 2006.

Published by the Debian Project [7].

Contents

List of Figures

List of Tables

Preface

I never meant to write this book. It emerged unheralded, unexpectedly.

The book began in 1983 when a high-school classmate challenged me to prove the Pythagorean theorem on the spot. I lost the dare, but looking the proof up later I recorded it on loose leaves, adding to it the derivations of a few other theorems of interest to me. From such a kernel the notes grew over time, until family and friends suggested that the notes might make the material for a book.

The book, a work yet in progress but a complete, entire book as it stands, first frames coherently the simplest, most basic derivations of applied mathematics, treating quadratics, trigonometrics, exponentials, derivatives, integrals, series, complex variables and, of course, the aforementioned Pythagorean theorem. These and others establish the book's foundation in Chs. 2 through 9. Later chapters build upon the foundation, deriving results less general or more advanced. Such is the book's plan.

A book can follow convention or depart from it; yet, though occasional departure might render a book original, frequent departure seldom renders a book good. Whether this particular book is original or good, neither or both, is for the reader to tell, but in any case the book does both follow and depart. Convention is a peculiar thing: at its best, it evolves or accumulates only gradually, patiently storing up the long, hidden wisdom of generations past; yet herein arises the ancient dilemma. Convention, in all its richness, in all its profundity, can, sometimes, stagnate at a local maximum, a hillock whence higher ground is achievable not by gradual ascent but only by descent first—or by a leap. Descent risks a bog. A leap risks a fall. One ought not run such risks without cause, even in such an inherently unconservative discipline as mathematics.

Well, the book does risk. It risks one leap at least: it employs hexadecimal numerals.

Decimal numerals are fine in history and anthropology (man has ten fingers), finance and accounting (dollars, cents, pounds, shillings, pence: the base hardly matters), law and engineering (the physical units are arbitrary anyway); but they are merely serviceable in mathematical theory, never aesthetic. There unfortunately really is no gradual way to bridge the gap to hexadecimal (shifting to base eleven, thence to twelve, etc., is no use). If one wishes to reach hexadecimal ground, one must leap. Twenty years of keeping my own private notes in hex have persuaded me that the leap justifies the risk. In other matters, by contrast, the book leaps seldom. The book in general walks a tolerably conventional applied mathematical line.

The book belongs to the emerging tradition of open-source software, where at the time of this writing it fills a void. Nevertheless it is a *book,* not a program. Lore among open-source developers holds that open development inherently leads to superior work. Well, maybe. Often it does in fact. Personally with regard to my own work, I should rather not make too many claims. It would be vain to deny that professional editing and formal peer review, neither of which the book enjoys, had substantial value. On the other hand, it does not do to despise the *amateur* (literally, one who does for the love of it: not such a bad motive, after all[1]) on principle, either—unless one would on the same principle despise a Washington or an Einstein, or a Debian Developer [7]. Open source has a spirit to it which leads readers to be far more generous with their feedback than ever could be the case with a traditional, proprietary book. Such readers, among whom a surprising concentration of talent and expertise are found, enrich the work freely. This has value, too.

The book's peculiar mission and program lend it an unusual quantity of discursive footnotes. These footnotes offer nonessential material which, while edifying, coheres insufficiently well to join the main narrative. The footnote is an imperfect messenger, of course. Catching the reader's eye, it can break the flow of otherwise good prose. Modern publishing offers various alternatives to the footnote—numbered examples, sidebars, special fonts, colored inks, etc. Some of these are merely trendy. Others, like numbered examples, really do help the right kind of book; but for this book the humble footnote, long sanctioned by an earlier era of publishing, extensively employed by such sages as Gibbon [12] and Shirer [23], seems the most able messenger. In this book it shall have many messages to bear.

The book provides a bibliography listing other books I have referred to while writing. Mathematics by its very nature promotes queer bibliographies, however, for its methods and truths are established by derivation rather than authority. Much of the book consists of common mathematical

[1] The expression is derived from an observation I seem to recall George F. Will making.

knowledge or of proofs I have worked out with my own pencil from various ideas gleaned who knows where over the years. The latter proofs are perhaps original or semi-original from my personal point of view, but it is unlikely that many if any of them are truly new. To the initiated, the mathematics itself often tends to suggest the form of the proof: if to me, then surely also to others who came before; and even where a proof is new the idea proven probably is not.

Some of the books in the bibliography are indeed quite good, but you should not necessarily interpret inclusion as more than a source acknowledgment by me. They happen to be books I have on my own bookshelf for whatever reason (had bought it for a college class years ago, had found it once at a yard sale for 25 cents, etc.), or have borrowed, in which I looked something up while writing.

As to a grand goal, underlying purpose or hidden plan, the book has none, other than to derive as many useful mathematical results as possible and to record the derivations together in an orderly manner in a single volume. What constitutes "useful" or "orderly" is a matter of perspective and judgment, of course. My own peculiar heterogeneous background in military service, building construction, electrical engineering, electromagnetic analysis and Debian development, my nativity, residence and citizenship in the United States, undoubtedly bias the selection and presentation to some degree. How other authors go about writing their books, I do not know, but I suppose that what is true for me is true for many of them also: we begin by organizing notes for our own use, then observe that the same notes may prove useful to others, and then undertake to revise the notes and to bring them into a form which actually is useful to others. Whether this book succeeds in the last point is for the reader to judge.

THB

Chapter 1

Introduction

This is a book of applied mathematical proofs. If you have seen a mathematical result, if you want to know why the result is so, you can look for the proof here.

The book's purpose is to convey the essential ideas underlying the derivations of a large number of mathematical results useful in the modeling of physical systems. To this end, the book emphasizes main threads of mathematical argument and the motivation underlying the main threads, deemphasizing formal mathematical rigor. It derives mathematical results from the purely applied perspective of the scientist and the engineer.

The book's chapters are topical. This first chapter treats a few introductory matters of general interest.

1.1 Applied mathematics

What is applied mathematics?

> Applied mathematics is a branch of mathematics that concerns itself with the application of mathematical knowledge to other domains... The question of what is applied mathematics does not answer to logical classification so much as to the sociology of professionals who use mathematics. [1]

That is about right, on both counts. In this book we shall define *applied mathematics* to be correct mathematics useful to scientists, engineers and the like; proceeding not from reduced, well defined sets of axioms but rather directly from a nebulous mass of natural arithmetical, geometrical and classical-algebraic idealizations of physical systems; demonstrable but generally lacking the detailed rigor of the professional mathematician.

1.2 Rigor

It is impossible to write such a book as this without some discussion of mathematical rigor. Applied and professional mathematics differ principally and essentially in the layer of abstract definitions the latter subimposes beneath the physical ideas the former seeks to model. Notions of mathematical rigor fit far more comfortably in the abstract realm of the professional mathematician; they do not always translate so gracefully to the applied realm. Of this difference, the applied mathematical reader and practitioner needs to be aware.

1.2.1 Axiom and definition

Ideally, a professional mathematician knows or precisely specifies in advance the set of fundamental axioms he means to use to derive a result. A prime aesthetic here is irreducibility: no axiom in the set should overlap the others or be specifiable in terms of the others. Geometrical argument—proof by sketch—is distrusted. The professional mathematical literature discourages undue pedantry indeed, but its readers do implicitly demand a convincing assurance that its writers *could* derive results in pedantic detail if called upon to do so. Precise definition here is critically important, which is why the professional mathematician tends not to accept blithe statements such as that

$$\frac{1}{0} = \infty,$$

without first inquiring as to exactly what is meant by symbols like 0 and ∞.

The applied mathematician begins from a different base. His ideal lies not in precise definition or irreducible axiom, but rather in the elegant modeling of the essential features of some physical system. Here, mathematical definitions tend to be made up *ad hoc* along the way, based on previous experience solving similar problems, adapted implicitly to suit the model at hand. If you ask the applied mathematician exactly what his axioms are, which symbolic algebra he is using, he usually doesn't know; what he knows is that the bridge has its footings in certain soils with specified tolerances, suffers such-and-such a wind load, etc. To avoid error, the applied mathematician relies not on abstract formalism but rather on a thorough mental grasp of the essential physical features of the phenomenon he is trying to model. An equation like

$$\frac{1}{0} = \infty$$

may make perfect sense without further explanation to an applied mathematical readership, depending on the physical context in which the equation is introduced. Geometrical argument—proof by sketch—is not only trusted but treasured. Abstract definitions are wanted only insofar as they smooth the analysis of the particular physical problem at hand; such definitions are seldom promoted for their own sakes.

The irascible Oliver Heaviside, responsible for the important applied mathematical technique of phasor analysis, once said,

> It is shocking that young people should be addling their brains over mere logical subtleties, trying to understand the proof of one obvious fact in terms of something equally ... obvious. [2]

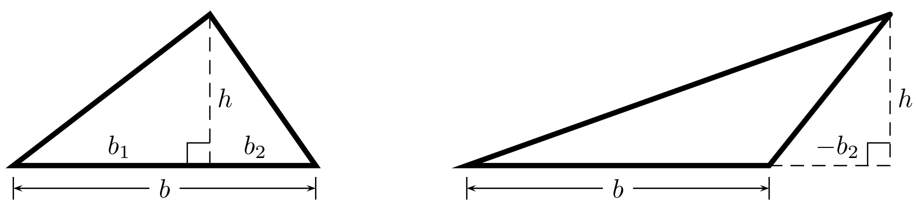
Exaggeration, perhaps, but from the applied mathematical perspective Heaviside nevertheless had a point. The professional mathematicians R. Courant and D. Hilbert put it more soberly in 1924 when they wrote,

> Since the seventeenth century, physical intuition has served as a vital source for mathematical problems and methods. Recent trends and fashions have, however, weakened the connection between mathematics and physics; mathematicians, turning away from the roots of mathematics in intuition, have concentrated on refinement and emphasized the postulational side of mathematics, and at times have overlooked the unity of their science with physics and other fields. In many cases, physicists have ceased to appreciate the attitudes of mathematicians... [6, Preface]

Although the present book treats "the attitudes of mathematicians" with greater deference than some of the unnamed 1924 physicists might have done, still, Courant and Hilbert could have been speaking for the engineers and other applied mathematicians of our own day as well as for the physicists of theirs. To the applied mathematician, the mathematics is not principally meant to be developed and appreciated for its own sake; it is meant to be *used.* This book adopts the Courant-Hilbert perspective.

The introduction you are now reading is not the right venue for an essay on why both kinds of mathematics—applied and professional (or pure)—are needed. Each kind has its place; and although it is a stylistic error to mix the two indiscriminately, clearly the two have much to do with one another. However this may be, this book is a book of derivations of applied mathematics. The derivations here proceed by a purely applied approach.

Figure 1.1: Two triangles.



1.2.2 Mathematical extension

Profound results in mathematics are occasionally achieved simply by extending results already known. For example, negative integers and their properties can be discovered by counting backward—3, 2, 1, 0—then asking what follows (precedes?) 0 in the countdown and what properties this new, negative integer must have to interact smoothly with the already known positives. The astonishing Euler's formula (§ 5.4) is discovered by a similar but more sophisticated mathematical extension.

More often, however, the results achieved by extension are unsurprising and not very interesting in themselves. Such extended results are the faithful servants of mathematical rigor. Consider for example the triangle on the left of Fig. 1.1. This triangle is evidently composed of two right triangles of areas

$$\begin{aligned} A_1 &= \frac{b_1 h}{2}, \\ A_2 &= \frac{b_2 h}{2} \end{aligned}$$

(each right triangle is exactly half a rectangle). Hence the main triangle's area is

$$A = A_1 + A_2 = \frac{(b_1 + b_2)h}{2} = \frac{bh}{2}.$$

Very well. What about the triangle on the right? Its b_1 is not shown on the figure, and what is that $-b_2$, anyway? Answer: the triangle is composed of the *difference* of two right triangles, with b_1 the base of the larger, overall one: $b_1 = b + (-b_2)$. The b_2 is negative because the sense of the small right triangle's area in the proof is negative: the small area is subtracted from the large rather than added. By extension on this basis, the main triangle's area is again seen to be $A = bh/2$. The proof is exactly the same. In fact, once the central idea of adding two right triangles is grasped, the extension is really rather obvious—too obvious to be allowed to burden such a book as this.

Excepting the uncommon cases where extension reveals something interesting or new, this book generally leaves the mere extension of proofs—including the validation of edge cases and over-the-edge cases—as an exercise to the interested reader.

1.3 Complex numbers and complex variables

More than a mastery of mere logical details, it is an holistic view of the mathematics and of its use in the modeling of physical systems which is the mark of the applied mathematician. A *feel* for the math is the great thing. Formal definitions, axioms, symbolic algebras and the like, though often useful, are felt to be secondary. The book's rapidly staged development of complex numbers and complex variables is planned on this sensibility.

Sections 2.12, 3.11, 4.3.3, 4.4, 6.2, 8.4, 8.5, 8.6 and 9.5, plus all of Chapter 5, constitute the book's principal stages of complex development. In these sections and throughout the book, the reader comes to appreciate that most mathematical properties which apply for real numbers apply equally for complex, that few properties concern real numbers alone.

1.4 On the text

The book gives numerals in hexadecimal. It denotes variables in Greek letters as well as Roman. Readers unfamiliar with the hexadecimal notation will find a brief orientation thereto in Appendix A. Readers unfamiliar with the Greek alphabet will find it in Appendix B.

Licensed to the public under the GNU General Public Licence [11], version 2, this book meets the Debian Free Software Guidelines [8].

If you cite an equation, section, chapter, figure or other item from this book, it is recommended that you include in your citation the book's precise draft date as given on the title page. The reason is that equation numbers, chapter numbers and the like are numbered automatically by the LaTeX typesetting software: such numbers can change arbitrarily from draft to draft. If an example citation helps, see [5] in the bibliography.

Chapter 2

Classical algebra and geometry

Arithmetic and the simplest elements of classical algebra and geometry, we learn as children. Few readers will want this book to begin with a treatment of $1 + 1 = 2$; or of how to solve $3x - 2 = 7$. However, there are some basic points which do seem worth touching. The book starts with these.

2.1 Basic arithmetic relationships

This section states some arithmetical rules.

2.1.1 Commutivity, associativity, distributivity, identity and inversion

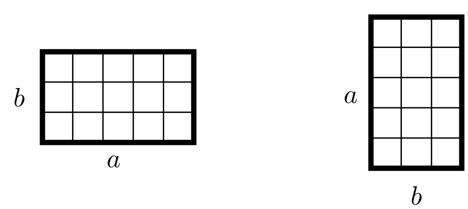
Refer to Table 2.1, whose rules apply equally to real and complex numbers (§ 2.12). Most of the rules are appreciated at once if the meaning of the symbols is understood. In the case of multiplicative commutivity, one imagines a rectangle with sides of lengths a and b, then the same rectangle turned on its side, as in Fig. 2.1: since the area of the rectangle is the same in either case, and since the area is the length times the width in either case (the area is more or less a matter of counting the little squares), evidently multiplicative commutivity holds. A similar argument validates multiplicative associativity, except that here we compute the *volume* of a three-dimensional rectangular box, which box we turn various ways.

Multiplicative inversion lacks an obvious interpretation when $a = 0$.

Table 2.1: Basic properties of arithmetic.

$a+b$	$=$	$b+a$	Additive commutivity
$a+(b+c)$	$=$	$(a+b)+c$	Additive associativity
$a+0=0+a$	$=$	a	Additive identity
$a+(-a)$	$=$	0	Additive inversion
ab	$=$	ba	Multiplicative commutivity
$(a)(bc)$	$=$	$(ab)(c)$	Multiplicative associativity
$(a)(1)=(1)(a)$	$=$	a	Multiplicative identity
$(a)(1/a)$	$=$	1	Multiplicative inversion
$(a)(b+c)$	$=$	$ab+ac$	Distributivity

Figure 2.1: Multiplicative commutivity.



Loosely,

$$\frac{1}{0} = \infty.$$

But since $3/0 = \infty$ also, surely either the zero or the infinity, or both, somehow differ in the latter case.

Looking ahead in the book, we note that the multiplicative properties do not always hold for more general linear transformations. For example, matrix multiplication is not commutative and vector cross-multiplication is not associative. Where associativity does not hold and parentheses do not otherwise group, right-to-left association is notationally implicit:[1]

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

The sense of it is that the thing on the left ($\mathbf{A}\times$) *operates* on the thing on the right ($\mathbf{B}\times\mathbf{C}$). (In the rare case where the question arises, you may want to use parentheses anyway.)

(Reference: [24, Ch. 1].)

2.1.2 Negative numbers

Consider that

$$\begin{aligned}
(+a)(+b) &= +ab, \\
(+a)(-b) &= -ab, \\
(-a)(+b) &= -ab, \\
(-a)(-b) &= +ab.
\end{aligned}$$

The first three of the four equations are unsurprising, but the last is interesting. Why would a negative count $-a$ of a negative quantity $-b$ come to

[1] The fine C and C++ programming languages are unfortunately stuck with the reverse order of association, along with division inharmoniously on the same level of syntactic precedence as multiplication. Standard mathematical notation is more elegant:

$$abc/uvw = \frac{(a)(bc)}{(u)(vw)}.$$

a positive product $+ab$? To see why, consider the progression

$$\begin{array}{rcr} & \vdots & \\ (+3)(-b) & = & -3b, \\ (+2)(-b) & = & -2b, \\ (+1)(-b) & = & -1b, \\ (0)(-b) & = & 0b, \\ (-1)(-b) & = & +1b, \\ (-2)(-b) & = & +2b, \\ (-3)(-b) & = & +3b, \\ & \vdots & \end{array}$$

The logic of arithmetic demands that the product of two negative numbers be positive for this reason.

2.1.3 Inequality

If[2]

$$a < b,$$

this necessarily implies that

$$a + x < b + x.$$

However, the relationship between ua and ub depends on the sign of u:

$$\begin{array}{ll} ua < ub & \text{if } u > 0; \\ ua > ub & \text{if } u < 0. \end{array}$$

Also,

$$\frac{1}{a} > \frac{1}{b}.$$

[2]Few readers attempting this book will need to be reminded that $<$ means "is less than," that $>$ means "is greater than," or that $\leq$ and $\geq$ respectively mean "is less than or equal to" and "is greater than or equal to."

2.1.4 The change of variable

The applied mathematician very often finds it convenient to *change variables,* introducing new symbols to stand in place of old. For this we have the *change of variable* or *assignment* notation[3]

$$Q \leftarrow P.$$

This means, "in place of P, put Q;" or, "let Q now equal P." For example, if $a^2 + b^2 = c^2$, then the change of variable $2\mu \leftarrow a$ yields the new form $(2\mu)^2 + b^2 = c^2$.

Similar to the change of variable notation is the definition notation

$$Q \equiv P.$$

This means, "let the new symbol Q represent P."[4]

The two notations logically mean about the same thing. Subjectively, $Q \equiv P$ identifies a quantity P sufficiently interesting to be given a permanent name Q, whereas $Q \leftarrow P$ implies nothing especially interesting about P or Q; it just introduces a (perhaps temporary) new symbol Q to ease the algebra. The concepts grow clearer as examples of the usage arise in the book.

2.2 Quadratics

It is often convenient to factor differences and sums of squares as

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b), \\ a^2 + b^2 &= (a+ib)(a-ib), \\ a^2 - 2ab + b^2 &= (a-b)^2, \\ a^2 + 2ab + b^2 &= (a+b)^2 \end{aligned} \tag{2.1}$$

[3]There appears to exist no broadly established standard mathematical notation for the change of variable, other than the = equal sign, which regrettably does not fill the role well. One can indeed use the equal sign, but then what does the change of variable $k = k + 1$ mean? It looks like a claim that k and $k + 1$ are the same, which is impossible. The notation $k \leftarrow k + 1$ by contrast is unambiguous; it means to increment k by one. However, the latter notation admittedly has seen only scattered use in the literature.

The C and C++ programming languages use `==` for equality and `=` for assignment (change of variable), as the reader may be aware.

[4]One would never write $k \equiv k + 1$. Even $k \leftarrow k + 1$ can confuse readers inasmuch as it appears to imply two different values for the same symbol k, but the latter notation is sometimes used anyway when new symbols are unwanted or because more precise alternatives (like $k_n = k_{n-1} + 1$) seem overwrought. Still, usually it is better to introduce a new symbol, as in $j \leftarrow k + 1$.

In some books, $\equiv$ is printed as $\stackrel{\triangle}{=}$.

(where i is the *imaginary unit,* a number defined such that $i^2 = -1$, introduced in more detail in § 2.12 below). Useful as these four forms are, however, none of them can directly factor a more general quadratic[5] expression like

$$z^2 - 2\beta z + \gamma^2.$$

To factor this, we *complete the square,* writing

$$\begin{aligned} z^2 - 2\beta z + \gamma^2 &= z^2 - 2\beta z + \gamma^2 + (\beta^2 - \gamma^2) - (\beta^2 - \gamma^2) \\ &= z^2 - 2\beta z + \beta^2 - (\beta^2 - \gamma^2) \\ &= (z - \beta)^2 - (\beta^2 - \gamma^2). \end{aligned}$$

The expression evidently has roots[6] where

$$(z - \beta)^2 = (\beta^2 - \gamma^2),$$

or in other words where

$$z = \beta \pm \sqrt{\beta^2 - \gamma^2}. \tag{2.2}$$

This suggests the factoring[7]

$$z^2 - 2\beta z + \gamma^2 = (z - z_1)(z - z_2), \tag{2.3}$$

where z_1 and z_2 are the two values of z given by (2.2).

It follows that the two solutions of the quadratic equation

$$z^2 = 2\beta z - \gamma^2 \tag{2.4}$$

are those given by (2.2), which is called *the quadratic formula.*[8] (*Cubic* and *quartic formulas* also exist respectively to extract the roots of polynomials of third and fourth order, but they are much harder. See Ch. 10 and its Tables 10.1 and 10.2.)

[5]The adjective *quadratic* refers to the algebra of expressions in which no term has greater than second order. Examples of quadratic expressions include x^2, $2x^2 - 7x + 3$ and $x^2 + 2xy + y^2$. By contrast, the expressions $x^3 - 1$ and $5x^2y$ are *cubic* not quadratic because they contain third-order terms. First-order expressions like $x + 1$ are *linear;* zeroth-order expressions like 3 are *constant.* Expressions of fourth and fifth order are *quartic* and *quintic,* respectively. (If not already clear from the context, *order* basically refers to the number of variables multiplied together in a term. The term $5x^2y = 5(x)(x)(y)$ is of third order, for instance.)

[6]A *root* of $f(z)$ is a value of z for which $f(z) = 0$.

[7]It suggests it because the expressions on the left and right sides of (2.3) are both quadratic (the highest power is z^2) and have the same roots. Substituting into the equation the values of z_1 and z_2 and simplifying proves the suggestion correct.

[8]The form of the quadratic formula which usually appears in print is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

2.3 Notation for series sums and products

Sums and products of series arise so frequently in mathematical work that one finds it convenient to define terse notations to express them. The summation notation

$$\sum_{k=a}^{b} f(k)$$

means to let k equal each of $a, a+1, a+2, \ldots, b$ in turn, evaluating the function $f(k)$ at each k, then adding the several $f(k)$. For example,[9]

$$\sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 = \mathrm{0x56}.$$

The similar multiplication notation

$$\prod_{k=a}^{b} f(k)$$

means to *multiply* the several $f(k)$ rather than to add them. The symbols $\sum$ and $\prod$ come respectively from the Greek letters for S and P, and may be regarded as standing for "Sum" and "Product." The k is a *dummy variable, index of summation* or *loop counter*—a variable with no independent existence, used only to facilitate the addition or multiplication of the series.[10]

The product shorthand

$$\begin{aligned} n! &\equiv \prod_{k=1}^{n} k, \\ n!/m! &\equiv \prod_{k=m+1}^{n} k, \end{aligned}$$

---

which solves the quadratic $ax^2 + bx + c = 0$. However, this writer finds the form (2.2) easier to remember. For example, by (2.2) in light of (2.4), the quadratic

$$z^2 = 3z - 2$$

has the solutions

$$z = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2} = 1 \text{ or } 2.$$

[9]The hexadecimal numeral 0x56 represents the same number the decimal numeral 86 represents. The book's preface explains why the book represents such numbers in hexadecimal. Appendix A tells how to read the numerals.

[10]Section 7.3 speaks further of the dummy variable.

is very frequently used. The notation $n!$ is pronounced "n factorial." Regarding the notation $n!/m!$, this can of course be regarded correctly as $n!$ divided by $m!$, but it usually proves more amenable to regard the notation as a single unit.[11]

Because multiplication in its more general sense as linear transformation is not always commutative, we specify that

$$\prod_{k=a}^{b} f(k) = [f(b)][f(b-1)][f(b-2)]\cdots[f(a+2)][f(a+1)][f(a)]$$

rather than the reverse order of multiplication.[12] Multiplication proceeds from right to left. In the event that the reverse order of multiplication is needed, we shall use the notation

$$\coprod_{k=a}^{b} f(k) = [f(a)][f(a+1)][f(a+2)]\cdots[f(b-2)][f(b-1)][f(b)].$$

Note that for the sake of definitional consistency,

$$\begin{aligned}
\sum_{k=N+1}^{N} f(k) &= 0 + \sum_{k=N+1}^{N} f(k) = 0,\\
\prod_{k=N+1}^{N} f(k) &= (1) \prod_{k=N+1}^{N} f(k) = 1.
\end{aligned}$$

This means among other things that

$$0! = 1. \tag{2.5}$$

On first encounter, such $\sum$ and $\prod$ notation seems a bit overwrought. Admittedly it is easier for the beginner to read "$f(1)+f(2)+\cdots+f(N)$" than "$\sum_{k=1}^{N} f(k)$." However, experience shows the latter notation to be extremely useful in expressing more sophisticated mathematical ideas. We shall use such notation extensively in this book.

[11] One reason among others for this is that factorials rapidly multiply to extremely large sizes, overflowing computer registers during numerical computation. If you can avoid unnecessary multiplication by regarding $n!/m!$ as a single unit, this is a win.

[12] The extant mathematical literature lacks an established standard on the order of multiplication implied by the "$\prod$" symbol, but this is the order we shall use in this book.

2.4 The arithmetic series

A simple yet useful application of the series sum of § 2.3 is the *arithmetic series*

$$\sum_{k=a}^{b} k = a + (a+1) + (a+2) + \cdots + b.$$

Pairing a with b, then $a+1$ with $b-1$, then $a+2$ with $b-2$, etc., the average of each pair is $[a+b]/2$; thus the average of the entire series is $[a+b]/2$. (The pairing may or may not leave an unpaired element at the series midpoint $k = [a+b]/2$, but this changes nothing.) The series has $b - a + 1$ terms. Hence,

$$\sum_{k=a}^{b} k = (b - a + 1)\frac{a+b}{2}. \tag{2.6}$$

Success with this arithmetic series leads one to wonder about the *geometric series* $\sum_{k=0}^{\infty} z^k$. Section 2.6.3 addresses that point.

2.5 Powers and roots

This necessarily tedious section discusses powers and roots. It offers no surprises. Table 2.2 summarizes its definitions and results. Readers seeking more rewarding reading may prefer just to glance at the table then to skip directly to the start of the next section.

In this section, the exponents k, m, n, p, q, r and s are integers,[13] but the exponents a and b are arbitrary real numbers.

2.5.1 Notation and integral powers

The power notation

$$z^n$$

[13]In case the word is unfamiliar to the reader who has learned arithmetic in another language than English, the *integers* are the negative, zero and positive counting numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ The corresponding adjective is *integral* (although the word "integral" is also used as a noun and adjective indicating an infinite sum of infinitesimals; see Ch. 7). Traditionally, the letters i, j, k, m, n, M and N are used to represent integers (i is sometimes avoided because the same letter represents the imaginary unit), but this section needs more integer symbols so it uses p, q, r and s, too.

Table 2.2: Power properties and definitions.

$$
\begin{aligned}
z^n &\equiv \prod_{k=1}^{n} z, \quad z \geq 0 \\
z &= (z^{1/n})^n = (z^n)^{1/n} \\
\sqrt{z} &\equiv z^{1/2} \\
(uv)^a &= u^a v^a \\
z^{p/q} &= (z^{1/q})^p = (z^p)^{1/q} \\
z^{ab} &= (z^a)^b = (z^b)^a \\
z^{a+b} &= z^a z^b \\
z^{a-b} &= \frac{z^a}{z^b} \\
z^{-b} &= \frac{1}{z^b}
\end{aligned}
$$

indicates the number z, multiplied by itself n times. More formally, when the *exponent* n is a nonnegative integer,[14]

$$z^n \equiv \prod_{k=1}^{n} z. \tag{2.7}$$

For example,[15]

$$
\begin{aligned}
z^3 &= (z)(z)(z), \\
z^2 &= (z)(z), \\
z^1 &= z, \\
z^0 &= 1.
\end{aligned}
$$

[14]The symbol "$\equiv$" means "$=$", but it further usually indicates that the expression on its right serves to define the expression on its left. Refer to § 2.1.4.

[15]The case 0^0 is interesting because it lacks an obvious interpretation. The specific interpretation depends on the nature and meaning of the two zeros. For interest, if $E \equiv 1/\epsilon$, then

$$\lim_{\epsilon \to 0^+} \epsilon^\epsilon = \lim_{E \to \infty} \left(\frac{1}{E}\right)^{1/E} = \lim_{E \to \infty} E^{-1/E} = \lim_{E \to \infty} e^{-(\ln E)/E} = e^0 = 1.$$

Notice that in general,

$$z^{n-1} = \frac{z^n}{z}.$$

This leads us to extend the definition to negative integral powers with

$$z^{-n} = \frac{1}{z^n}. \tag{2.8}$$

From the foregoing it is plain that

$$\begin{aligned} z^{m+n} &= z^m z^n, \\ z^{m-n} &= \frac{z^m}{z^n}, \end{aligned} \tag{2.9}$$

for any integral m and n. For similar reasons,

$$z^{mn} = (z^m)^n = (z^n)^m. \tag{2.10}$$

On the other hand from multiplicative associativity and commutivity,

$$(uv)^n = u^n v^n. \tag{2.11}$$

2.5.2 Roots

Fractional powers are not something we have defined yet, so for consistency with (2.10) we let

$$(u^{1/n})^n = u.$$

This has $u^{1/n}$ as the number which, raised to the nth power, yields u. Setting

$$v = u^{1/n},$$

it follows by successive steps that

$$\begin{aligned} v^n &= u, \\ (v^n)^{1/n} &= u^{1/n}, \\ (v^n)^{1/n} &= v. \end{aligned}$$

Taking the u and v formulas together, then,

$$(z^{1/n})^n = z = (z^n)^{1/n} \tag{2.12}$$

for any z and integral n.

The number $z^{1/n}$ is called the *nth root* of z—or in the very common case $n = 2$, the *square root* of z, often written

$$\sqrt{z}.$$

When z is real and nonnegative, the last notation is usually implicitly taken to mean the real, nonnegative square root. In any case, the power and root operations mutually invert one another.

What about powers expressible neither as n nor as $1/n$, such as the $3/2$ power? If z and w are numbers related by

$$w^q = z,$$

then

$$w^{pq} = z^p.$$

Taking the qth root,

$$w^p = (z^p)^{1/q}.$$

But $w = z^{1/q}$, so this is

$$(z^{1/q})^p = (z^p)^{1/q},$$

which says that it does not matter whether one applies the power or the root first; the result is the same. Extending (2.10) therefore, we define $z^{p/q}$ such that

$$(z^{1/q})^p = z^{p/q} = (z^p)^{1/q}. \tag{2.13}$$

Since any real number can be approximated arbitrarily closely by a ratio of integers, (2.13) implies a power definition for all real exponents.

Equation (2.13) is this subsection's main result. However, § 2.5.3 will find it useful if we can also show here that

$$(z^{1/q})^{1/s} = z^{1/qs} = (z^{1/s})^{1/q}. \tag{2.14}$$

The proof is straightforward. If

$$w \equiv z^{1/qs},$$

then raising to the qs power yields

$$(w^s)^q = z.$$

Successively taking the qth and sth roots gives

$$w = (z^{1/q})^{1/s}.$$

By identical reasoning,

$$w = (z^{1/s})^{1/q}.$$

But since $w \equiv z^{1/qs}$, the last two equations imply (2.14), as we have sought.

2.5.3 Powers of products and powers of powers

Per (2.11),

$$(uv)^p = u^p v^p.$$

Raising this equation to the $1/q$ power, we have that

$$\begin{aligned}
(uv)^{p/q} &= [u^p v^p]^{1/q} \\
&= \left[(u^p)^{q/q}(v^p)^{q/q}\right]^{1/q} \\
&= \left[(u^{p/q})^q(v^{p/q})^q\right]^{1/q} \\
&= \left[(u^{p/q})(v^{p/q})\right]^{q/q} \\
&= u^{p/q}v^{p/q}.
\end{aligned}$$

In other words

$$(uv)^a = u^a v^a \tag{2.15}$$

for any real a.

On the other hand, per (2.10),

$$z^{pr} = (z^p)^r.$$

Raising this equation to the $1/qs$ power and applying (2.10), (2.13) and (2.14) to reorder the powers, we have that

$$z^{(p/q)(r/s)} = (z^{p/q})^{r/s}.$$

By identical reasoning,

$$z^{(p/q)(r/s)} = (z^{r/s})^{p/q}.$$

Since p/q and r/s can approximate any real numbers with arbitrary precision, this implies that

$$(z^a)^b = z^{ab} = (z^b)^a \tag{2.16}$$

for any real a and b.

2.5.4 Sums of powers

With (2.9), (2.15) and (2.16), one can reason that

$$z^{(p/q)+(r/s)} = (z^{ps+rq})^{1/qs} = (z^{ps}z^{rq})^{1/qs} = z^{p/q}z^{r/s},$$

or in other words that

$$z^{a+b} = z^a z^b. \tag{2.17}$$

In the case that $a = -b$,

$$1 = z^{-b+b} = z^{-b} z^b,$$

which implies that

$$z^{-b} = \frac{1}{z^b}. \tag{2.18}$$

But then replacing $b \leftarrow -b$ in (2.17) leads to

$$z^{a-b} = z^a z^{-b},$$

which according to (2.18) is

$$z^{a-b} = \frac{z^a}{z^b}. \tag{2.19}$$

2.5.5 Summary and remarks

Table 2.2 on page 16 summarizes this section's definitions and results.

Looking ahead to § 2.12, § 3.11 and Ch. 5, we observe that nothing in the foregoing analysis requires the base variables z, w, u and v to be real numbers; if complex (§ 2.12), the formulas remain valid. Still, the analysis does imply that the various exponents m, n, p/q, a, b and so on are real numbers. This restriction, we shall remove later, purposely defining the action of a complex exponent to comport with the results found here. With such a definition the results apply not only for all bases but also for all exponents, real or complex.

2.6 Multiplying and dividing power series

A *power series*[16] is a weighted sum of integral powers:

$$A(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \tag{2.20}$$

where the several a_k are arbitrary constants. This section discusses the multiplication and division of power series.

[16] Another name for the *power series* is *polynomial*. The word "polynomial" usually connotes a power series with a finite number of terms, but the two names in fact refer to essentially the same thing.

2.6.1 Multiplying power series

Given two power series

$$\begin{aligned} A(z) &= \sum_{k=-\infty}^{\infty} a_k z^k, \\ B(z) &= \sum_{k=-\infty}^{\infty} b_k z^k, \end{aligned} \tag{2.21}$$

the product of the two series is evidently

$$P(z) \equiv A(z)B(z) = \sum_{k=-\infty}^{\infty} \left[\left(\sum_{j=-\infty}^{\infty} a_j b_{k-j} \right) z^k \right]. \tag{2.22}$$

2.6.2 Dividing power series

The quotient $Q(z) = B(z)/A(z)$ of two power series is a little harder to calculate. The calculation is by long division. For example,

$$\begin{aligned} \frac{2z^2 - 3z + 3}{z - 2} &= \frac{2z^2 - 4z}{z - 2} + \frac{z + 3}{z - 2} = 2z + \frac{z + 3}{z - 2} \\ &= 2z + \frac{z - 2}{z - 2} + \frac{5}{z - 2} = 2z + 1 + \frac{5}{z - 2}. \end{aligned}$$

The strategy is to take the dividend[17] $B(z)$ piece by piece, purposely choosing pieces easily divided by $A(z)$.

If you feel that you understand the example, then that is really all there is to it, and you can skip the rest of the subsection if you like. One sometimes wants to express the long division of power series more formally, however. That is what the rest of the subsection is about.

Formally, we prepare the long division $B(z)/A(z)$ by writing

$$B(z) = A(z)Q_n(z) + R_n(z), \tag{2.23}$$

where $R_n(z)$ is a *remainder* (being the part of $B(z)$ *remaining* to be divided);

[17] If $Q(z)$ is a *quotient* and $R(z)$ a *remainder,* then $B(z)$ is a *dividend* (or *numerator*) and $A(z)$ a *divisor* (or *denominator*). Such are the Latin-derived names of the parts of a long division.

and

$$\begin{aligned}
A(z) &= \sum_{k=-\infty}^{K} a_k z^k, \quad a_K \neq 0, \\
B(z) &= \sum_{k=-\infty}^{N} b_k z^k, \\
R_N(z) &= B(z), \\
Q_N(z) &= 0, \\
R_n(z) &= \sum_{k=-\infty}^{n} r_{nk} z^k, \\
Q_n(z) &= \sum_{k=n-K+1}^{N-K} q_k z^k,
\end{aligned} \tag{2.24}$$

where K and N identify the greatest orders k of z^k present in $A(z)$ and $B(z)$, respectively.

Well, that is a lot of symbology. What does it mean? The key to understanding it lies in understanding (2.23), which is not one but several equations—one equation for each value of n, where $n = N, N-1, N-2, \ldots$ The dividend $B(z)$ and the divisor $A(z)$ stay the same from one n to the next, but the quotient $Q_n(z)$ and the remainder $R_n(z)$ change. At the start, $Q_N(z) = 0$ while $R_N(z) = B(z)$, but the thrust of the long division process is to build $Q_n(z)$ up by wearing $R_n(z)$ down. The goal is to grind $R_n(z)$ away to nothing, to make it disappear as $n \to -\infty$.

As in the example, we pursue the goal by choosing from $R_n(z)$ an easily divisible piece containing the whole high-order term of $R_n(z)$. The piece we choose is $(r_{nn}/a_K)z^{n-K}A(z)$, which we add and subtract from (2.23) to obtain

$$B(z) = A(z)\left[Q_n(z) + \frac{r_{nn}}{a_K}z^{n-K}\right] + \left[R_n(z) - \frac{r_{nn}}{a_K}z^{n-K}A(z)\right].$$

Matching this equation against the desired iterate

$$B(z) = A(z)Q_{n-1}(z) + R_{n-1}(z)$$

and observing from the definition of $Q_n(z)$ that $Q_{n-1}(z) = Q_n(z) + q_{n-K}z^{n-K}$, we find that

$$\begin{aligned}
q_{n-K} &= \frac{r_{nn}}{a_K}, \\
R_{n-1}(z) &= R_n(z) - q_{n-K}z^{n-K}A(z),
\end{aligned} \tag{2.25}$$

where no term remains in $R_{n-1}(z)$ higher than a z^{n-1} term.

To begin the actual long division, we initialize

$$R_N(z) = B(z),$$

for which (2.23) is trivially true. Then we iterate per (2.25) as many times as desired. If an infinite number of times, then so long as $R_n(z)$ tends to vanish as $n \to -\infty$, it follows from (2.23) that

$$\frac{B(z)}{A(z)} = Q_{-\infty}(z). \tag{2.26}$$

Iterating only a finite number of times leaves a remainder,

$$\frac{B(z)}{A(z)} = Q_n(z) + \frac{R_n(z)}{A(z)}, \tag{2.27}$$

except that it may happen that $R_n(z) = 0$ for sufficiently small n.

Table 2.3 summarizes the long-division procedure.

It should be observed in light of Table 2.3 that if

$$\begin{aligned} A(z) &= \sum_{k=K_o}^{K} a_k z^k, \\ B(z) &= \sum_{k=N_o}^{N} b_k z^k, \end{aligned}$$

then

$$R_n(z) = \sum_{k=n-(K-K_o)+1}^{n} r_{nk} z^k \quad \text{for all } n < N_o + (K - K_o). \tag{2.28}$$

That is, the remainder has order one less than the divisor has. The reason for this, of course, is that we have strategically planned the long-division iteration precisely to cause the leading term of the divisor to cancel the leading term of the remainder at each step.[18]

[18]If a more formal demonstration of (2.28) is wanted, then consider per (2.25) that

$$R_{m-1}(z) = R_m(z) - \frac{r_{mm}}{a_K} z^{m-K} A(z).$$

If the least-order term of $R_m(z)$ is a z^{N_o} term (as clearly is the case at least for the initial remainder $R_N(z) = B(z)$), then according to the equation so also must the least-

Table 2.3: Dividing power series through successively smaller powers.

$$
\begin{aligned}
B(z) &= A(z)Q_n(z) + R_n(z) \\
A(z) &= \sum_{k=-\infty}^{K} a_k z^k, \quad a_K \neq 0 \\
B(z) &= \sum_{k=-\infty}^{N} b_k z^k \\
R_N(z) &= B(z) \\
Q_N(z) &= 0 \\
R_n(z) &= \sum_{k=-\infty}^{n} r_{nk} z^k \\
Q_n(z) &= \sum_{k=n-K+1}^{N-K} q_k z^k \\
q_{n-K} &= \frac{r_{nn}}{a_K} \\
R_{n-1}(z) &= R_n(z) - q_{n-K} z^{n-K} A(z) \\
\frac{B(z)}{A(z)} &= Q_{-\infty}(z)
\end{aligned}
$$

Table 2.4: Dividing power series through successively larger powers.

$$
\begin{aligned}
B(z) &= A(z)Q_n(z) + R_n(z) \\
A(z) &= \sum_{k=K}^{\infty} a_k z^k, \quad a_K \neq 0 \\
B(z) &= \sum_{k=N}^{\infty} b_k z^k \\
R_N(z) &= B(z) \\
Q_N(z) &= 0 \\
R_n(z) &= \sum_{k=n}^{\infty} r_{nk} z^k \\
Q_n(z) &= \sum_{k=N-K}^{n-K-1} q_k z^k \\
q_{n-K} &= \frac{r_{nn}}{a_K} \\
R_{n+1}(z) &= R_n(z) - q_{n-K} z^{n-K} A(z) \\
\frac{B(z)}{A(z)} &= Q_\infty(z)
\end{aligned}
$$

The long-division procedure of Table 2.3 extends the quotient $Q_n(z)$ through successively smaller powers of z. Often, however, one prefers to extend the quotient through successively *larger* powers of z, where a z^K term is $A(z)$'s term of *least* order. In this case, the long division goes by the complementary rules of Table 2.4.

(Reference: [26, § 3.2].)

---

order term of $R_{m-1}(z)$ be a z^{N_o} term, unless an even lower-order term be contributed by the product $z^{m-K}A(z)$. But that very product's term of least order is a $z^{m-(K-K_o)}$ term. Under these conditions, evidently the least-order term of $R_{m-1}(z)$ is a $z^{m-(K-K_o)}$ term when $m - (K - K_o) \le N_o$; otherwise a z^{N_o} term. This is better stated after the change of variable $n+1 \leftarrow m$: the least-order term of $R_n(z)$ is a $z^{n-(K-K_o)+1}$ term when $n < N_o + (K - K_o)$; otherwise a z^{N_o} term.

The greatest-order term of $R_n(z)$ is by definition a z^n term. So, in summary, when $n < N_o + (K - K_o)$, the terms of $R_n(z)$ run from $z^{n-(K-K_o)+1}$ through z^n, which is exactly the claim (2.28) makes.

2.6.3 Common power-series quotients and the geometric series

Frequently encountered power-series quotients, calculated by the long division of § 2.6.2 and/or verified by multiplying, include[19]

$$\frac{1}{1 \pm z} = \begin{cases} \displaystyle\sum_{k=0}^{\infty} (\mp)^k z^k, & |z| < 1; \\ \displaystyle -\sum_{k=-\infty}^{-1} (\mp)^k z^k, & |z| > 1. \end{cases} \tag{2.29}$$

Equation (2.29) almost incidentally answers a question which has arisen in § 2.4 and which often arises in practice: to what total does the infinite *geometric series* $\sum_{k=0}^{\infty} z^k$, $|z| < 1$, sum? Answer: it sums exactly to $1/(1-z)$. However, there is a simpler, more aesthetic way to demonstrate the same thing, as follows. Let

$$S \equiv \sum_{k=0}^{\infty} z^k, \quad |z| < 1.$$

Multiplying by z yields

$$zS \equiv \sum_{k=1}^{\infty} z^k.$$

Subtracting the latter equation from the former leaves

$$(1 - z)S = 1,$$

which, after dividing by $1 - z$, implies that

$$S \equiv \sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}, \quad |z| < 1, \tag{2.30}$$

as was to be demonstrated.

2.6.4 Variations on the geometric series

Besides being more aesthetic than the long division of § 2.6.2, the difference technique of § 2.6.3 permits one to extend the basic geometric series in

[19] The notation $|z|$ represents the magnitude of z. For example, $|5| = 5$, but also $|-5| = 5$.

several ways. For instance, the sum

$$S_1 \equiv \sum_{k=0}^{\infty} kz^k, \quad |z| < 1$$

(which arises in, among others, Planck's quantum blackbody radiation calculation [17]), we can compute as follows. We multiply the unknown S_1 by z, producing

$$zS_1 = \sum_{k=0}^{\infty} kz^{k+1} = \sum_{k=1}^{\infty} (k-1)z^k.$$

We then subtract zS_1 from S_1, leaving

$$(1-z)S_1 = \sum_{k=0}^{\infty} kz^k - \sum_{k=1}^{\infty} (k-1)z^k = \sum_{k=1}^{\infty} z^k = z\sum_{k=0}^{\infty} z^k = \frac{z}{1-z},$$

where we have used (2.30) to collapse the last sum. Dividing by $1-z$, we arrive at

$$S_1 \equiv \sum_{k=0}^{\infty} kz^k = \frac{z}{(1-z)^2}, \quad |z| < 1, \tag{2.31}$$

which was to be found.

Further series of the kind, such as $\sum_k k^2 z^k$, $\sum_k (k+1)(k)z^k$, $\sum_k k^3 z^k$, etc., can be calculated in like manner as the need for them arises.

2.7 Indeterminate constants, independent variables and dependent variables

Mathematical models use *indeterminate constants, independent variables* and *dependent variables.* The three are best illustrated by example. Consider the time t a sound needs to travel from its source to a distant listener:

$$t = \frac{\Delta r}{v_{\mathrm{sound}}},$$

where Δr is the distance from source to listener and v_{sound} is the speed of sound. Here, v_{sound} is an indeterminate constant (given particular atmospheric conditions, it doesn't vary), Δr is an independent variable, and t is a dependent variable. The model gives t as a function of Δr; so, if you tell the model how far the listener sits from the sound source, the model returns the time needed for the sound to propagate from one to the other.

Note that the abstract validity of the model does not necessarily depend on whether we actually know the right figure for v_{sound} (if I tell you that sound goes at 500 m/s, but later you find out that the real figure is 331 m/s, it probably doesn't ruin the theoretical part of your analysis; you just have to recalculate numerically). Knowing the figure is not the point. The point is that conceptually there preëxists some right figure for the indeterminate constant; that sound goes at some constant speed—whatever it is—and that we can calculate the delay in terms of this.

Although there exists a definite philosophical distinction between the three kinds of quantity, nevertheless it cannot be denied that which particular quantity is an indeterminate constant, an independent variable or a dependent variable often depends upon one's immediate point of view. The same model in the example would remain valid if atmospheric conditions were changing (v_{sound} would then be an independent variable) or if the model were used in designing a musical concert hall[20] to suffer a maximum acceptable sound time lag from the stage to the hall's back row (t would then be an independent variable; Δr, dependent). Occasionally we go so far as deliberately to change our point of view in mid-analysis, now regarding as an independent variable what a moment ago we had regarded as an indeterminate constant, for instance (a typical case of this arises in the solution of differential equations by the method of unknown coefficients, § 9.4). Such a shift of viewpoint is fine, so long as we remember that there is a difference between the three kinds of quantity and we keep track of which quantity is which kind to us at the moment.

The main reason it matters which symbol represents which of the three kinds of quantity is that in calculus, one analyzes how change in independent variables affects dependent variables as indeterminate constants remain fixed.

(Section 2.3 has introduced the dummy variable, which the present sec-

[20]Math books are funny about examples like this. Such examples remind one of the kind of calculation one encounters in a childhood arithmetic textbook, as of the quantity of air contained in an astronaut's helmet. One could calculate the quantity of water in a kitchen mixing bowl just as well, but astronauts' helmets are so much more interesting than bowls, you see.

The chance that the typical reader will ever specify the dimensions of a real musical concert hall is of course vanishingly small. However, it is the idea of the example that matters here, because the chance that the typical reader will ever specify *something* technical is quite large. Although sophisticated models with many factors and terms do indeed play a major role in engineering, the great majority of practical engineering calculations—for quick, day-to-day decisions where small sums of money and negligible risk to life are at stake—are done with models hardly more sophisticated than the one shown here.

tion's threefold taxonomy seems to exclude. However in fact, most dummy variables are just independent variables—a few are dependent variables—whose scope is restricted to a particular expression. Such a dummy variable does not seem very "independent," of course; but its dependence is on the operator controlling the expression, not on some other variable within the expression. Within the expression, the dummy variable fills the role of an independent variable; without, it fills no role because logically it does not exist there. Refer to §§ 2.3 and 7.3.)

2.8 Exponentials and logarithms

In § 2.5 we considered the power operation z^a, where per § 2.7 z is an independent variable and a is an indeterminate constant. There is another way to view the power operation, however. One can view it as the *exponential* operation

$$a^z,$$

where the variable z is in the exponent and the constant a is in the base.

2.8.1 The logarithm

The exponential operation follows the same laws the power operation follows, but because the variable of interest is now in the exponent rather than the base, the inverse operation is not the root but rather the *logarithm:*

$$\log_a(a^z) = z. \tag{2.32}$$

The logarithm $\log_a w$ answers the question, "What power must I raise a to, to get w?"

Raising a to the power of the last equation, we have that

$$a^{\log_a(a^z)} = a^z.$$

With the change of variable $w \leftarrow a^z$, this is

$$a^{\log_a w} = w. \tag{2.33}$$

Hence, the exponential and logarithmic operations mutually invert one another.

2.8.2 Properties of the logarithm

The basic properties of the logarithm include that

$$
\begin{aligned}
\log_a uv &= \log_a u + \log_a v, && (2.34)\\
\log_a \frac{u}{v} &= \log_a u - \log_a v, && (2.35)\\
\log_a(w^p) &= p \log_a w, && (2.36)\\
\log_b w &= \frac{\log_a w}{\log_a b}. && (2.37)
\end{aligned}
$$

Of these, (2.34) follows from the steps

$$
\begin{aligned}
(uv) &= (u)(v),\\
(a^{\log_a uv}) &= (a^{\log_a u})(a^{\log_a v}),\\
a^{\log_a uv} &= a^{\log_a u + \log_a v};
\end{aligned}
$$

and (2.35) follows by similar reasoning. Equation (2.36) follows from the steps

$$
\begin{aligned}
(w^p) &= (w)^p,\\
a^{\log_a(w^p)} &= (a^{\log_a w})^p,\\
a^{\log_a(w^p)} &= a^{p \log_a w}.
\end{aligned}
$$

Equation (2.37) follows from the steps

$$
\begin{aligned}
w &= b^{\log_b w},\\
\log_a w &= \log_a(b^{\log_b w}),\\
\log_a w &= \log_b w \log_a b.
\end{aligned}
$$

Among other purposes, (2.34) through (2.37) serve respectively to transform products to sums, quotients to differences, powers to products, and logarithms to differently based logarithms. Table 2.5 repeats the equations, summarizing the general properties of the logarithm.

2.9 Triangles and other polygons: simple facts

This section gives simple facts about triangles and other polygons.

Table 2.5: General properties of the logarithm.

$$\begin{aligned}
\log_a uv &= \log_a u + \log_a v \\
\log_a \frac{u}{v} &= \log_a u - \log_a v \\
\log_a(w^p) &= p \log_a w \\
\log_b w &= \frac{\log_a w}{\log_a b}
\end{aligned}$$

2.9.1 Triangle area

The area of a right triangle[21] is half the area of the corresponding rectangle, seen by splitting a rectangle down one of its diagonals into two equal right triangles. The fact that *any* triangle's area is half its base length times its height is seen by dropping a perpendicular from one point of the triangle to the opposite side (see Fig. 1.1 on page 4), splitting the triangle into two right triangles, for each of which the fact is true. In algebraic symbols this is written

$$A = \frac{bh}{2}, \tag{2.38}$$

where A stands for area, b for base length, and h for perpendicular height.

2.9.2 The triangle inequalities

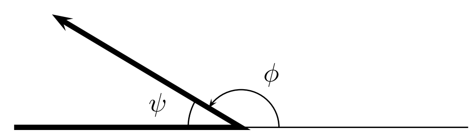
Any two sides of a triangle together are longer than the third alone, which itself is longer than the difference between the two. In symbols,

$$|a - b| < c < a + b, \tag{2.39}$$

where a, b and c are the lengths of a triangle's three sides. These are the *triangle inequalities.* The truth of the sum inequality $c < a + b$, is seen by sketching some triangle on a sheet of paper and asking: if c is the direct route between two points and $a + b$ is an indirect route, then how can $a + b$ not be longer? Of course the sum inequality is equally good on any of the triangle's three sides, so one can write $a < c + b$ and $b < c + a$ just as well as $c < a + b$. Rearranging the a and b inequalities, we have that $a - b < c$

[21] A *right triangle* is a triangle, one of whose three angles is perfectly square. Hence, dividing a rectangle down its diagonal produces a pair of right triangles.

Figure 2.2: The sum of a triangle's inner angles: turning at the corner.



and $b - a < c$, which together say that $|a - b| < c$. The last is the difference inequality, completing the proof of (2.39).

2.9.3 The sum of interior angles

A triangle's three interior angles[22] sum to $2\pi/2$. One way to see the truth of this fact is to imagine a small car rolling along one of the triangle's sides. Reaching the corner, the car turns to travel along the next side, and so on round all three corners to complete a circuit, returning to the start. Since the car again faces the original direction, we reason that it has turned a total of 2π: a full revolution. But the angle ϕ the car turns at a corner and the triangle's inner angle ψ there together form the straight angle $2\pi/2$ (the sharper the inner angle, the more the car turns: see Fig. 2.2). In mathematical notation,

$$\begin{aligned}
\phi_1 + \phi_2 + \phi_3 &= 2\pi, \\
\phi_k + \psi_k &= \frac{2\pi}{2}, \quad k = 1, 2, 3,
\end{aligned}$$

where ψ_k and ϕ_k are respectively the triangle's inner angles and the angles through which the car turns. Solving the latter equations for ϕ_k and

[22] Many or most readers will already know the notation 2π and its meaning as the angle of full revolution. For those who do not, the notation is introduced more properly in §§ 3.1, 3.6 and 8.9 below. Briefly, however, the symbol 2π represents a complete turn, a full circle, a spin to face the same direction as before. Hence $2\pi/4$ represents a square turn or right angle.

You may be used to the notation 360° in place of 2π, but for the reasons explained in Appendix A and in footnote 15 of Ch. 3, this book tends to avoid the former notation.

substituting into the former yields

$$\psi_1 + \psi_2 + \psi_3 = \frac{2\pi}{2}, \tag{2.40}$$

which was to be demonstrated.

Extending the same technique to the case of an n-sided polygon, we have that

$$\begin{aligned} \sum_{k=1}^{n} \phi_k &= 2\pi, \\ \phi_k + \psi_k &= \frac{2\pi}{2}. \end{aligned}$$

Solving the latter equations for ϕ_k and substituting into the former yields

$$\sum_{k=1}^{n} \left(\frac{2\pi}{2} - \psi_k \right) = 2\pi,$$

or in other words

$$\sum_{k=1}^{n} \psi_k = (n-2)\frac{2\pi}{2}. \tag{2.41}$$

Equation (2.40) is then seen to be a special case of (2.41) with $n = 3$.

2.10 The Pythagorean theorem

Along with Euler's formula (5.11) and the fundamental theorem of calculus (7.2), the Pythagorean theorem is perhaps one of the three most famous results in all of mathematics. The theorem holds that

$$a^2 + b^2 = c^2, \tag{2.42}$$

where a, b and c are the lengths of the legs and diagonal of a right triangle, as in Fig. 2.3. Many proofs of the theorem are known.

One such proof posits a square of side length $a+b$ with a tilted square of side length c inscribed as in Fig. 2.4. The area of each of the four triangles in the figure is evidently $ab/2$. The area of the tilted inner square is c^2. The area of the large outer square is $(a+b)^2$. But the large outer square is comprised of the tilted inner square plus the four triangles, hence the area

Figure 2.3: A right triangle.

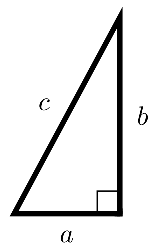
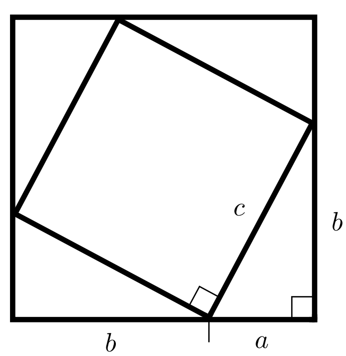


Figure 2.4: The Pythagorean theorem.



of the large outer square equals the area of the tilted inner square plus the areas of the four triangles. In mathematical symbols, this is

$$(a+b)^2 = c^2 + 4\left(\frac{ab}{2}\right),$$

which simplifies directly to (2.42).[23]

The Pythagorean theorem is readily extended to three dimensions as

$$a^2 + b^2 + h^2 = r^2, \tag{2.43}$$

where h is an altitude perpendicular to both a and b, thus also to c; and where r is the corresponding three-dimensional diagonal: the diagonal of the right triangle whose legs are c and h. Inasmuch as (2.42) applies to any right triangle, it follows that $c^2 + h^2 = r^2$, which equation expands directly to yield (2.43).

2.11 Functions

This book is not the place for a gentle introduction to the concept of the function. Briefly, however, a function is a mapping from one number (or vector of several numbers) to another. For example, $f(x) = x^2 - 1$ is a function which maps 1 to 0 and -3 to 8, among others.

When discussing functions, one often speaks of *domains* and *ranges.* The *domain* of a function is the set of numbers one can put into it. The *range* of a function is the corresponding set of numbers one can get out of it. In the example, if the domain is restricted to real x such that $|x| \le 3$, then the corresponding range is $-1 \le f(x) \le 8$.

Other terms which arise when discussing functions are *root* (or *zero*), *singularity* and *pole.* A *root* (or *zero*) of a function is a domain point at which the function evaluates to zero. In the example, there are roots at $x = \pm 1$. A *singularity* of a function is a domain point at which the function's

[23]This elegant proof is far simpler than the one famously given by the ancient geometer Euclid, yet more appealing than alternate proofs often found in print. Whether Euclid was acquainted with the simple proof given here this writer does not know, but it is possible [3, "Pythagorean theorem," 02:32, 31 March 2006] that Euclid chose his proof because it comported better with the restricted set of geometrical elements he permitted himself to work with. Be that as it may, the present writer encountered the proof this section gives somewhere years ago and has never seen it in print since, so can claim no credit for originating it. Unfortunately the citation is now long lost. A current source for the proof is [3] as cited above.

output *diverges;* that is, where the function's output is infinite.[24] A *pole* is a singularity which behaves locally like $1/x$ (rather than, for example, like $1/\sqrt{x}$). A singularity which behaves as $1/x^N$ is a *multiple pole*, which (§ 9.6.2) can be thought of as N poles. The example function $f(x)$ has no singularities for finite x; however, the function $h(x) = 1/(x^2 - 1)$ has poles at $x = \pm 1$.

(Besides the root, the singularity and the pole, there is also the troublesome *branch point,* an infamous example of which is $z = 0$ in the function $g(z) = \sqrt{z}$. Branch points are important, but the book must build a more extensive foundation before introducing them properly in § 8.5.)

2.12 Complex numbers (introduction)

Section 2.5.2 has introduced square roots. What it has not done is to tell us how to regard a quantity such as $\sqrt{-1}$. Since there exists no real number i such that

$$i^2 = -1 \tag{2.44}$$

and since the quantity i thus defined is found to be critically important across broad domains of higher mathematics, we accept (2.44) as the definition of a fundamentally new kind of quantity: *the imaginary number.*[25] Imaginary numbers are given their own number line, plotted at right angles to the familiar real number line as depicted in Fig. 2.5. The sum of a real number x and an imaginary number iy is the *complex number*

$$z = x + iy.$$

[24]Here is one example of the book's deliberate lack of formal mathematical rigor. A more precise formalism to say that "the function's output is infinite" might be

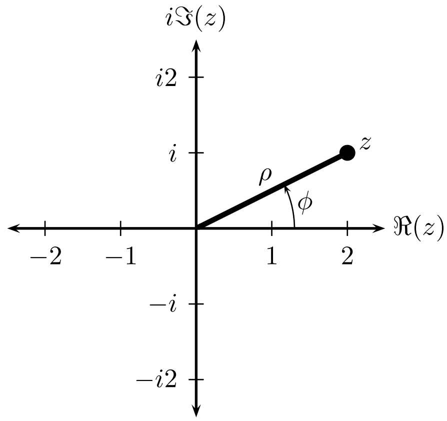
$$\lim_{x \to x_o} |f(x)| = \infty.$$

The applied mathematician tends to avoid such formalism where there seems no immediate use for it.

[25]The English word *imaginary* is evocative, but perhaps not evocative of quite the right concept here. Imaginary numbers are not to mathematics as imaginary elfs are to the physical world. In the physical world, imaginary elfs are (presumably) not substantial objects. However, in the mathematical realm, imaginary numbers *are* substantial. The word *imaginary* in the mathematical sense is thus more of a technical term than a descriptive adjective.

The reason imaginary numbers are called "imaginary" probably has to do with the fact that they emerge from mathematical operations only, never directly from counting or measuring physical things.

Figure 2.5: The complex (or Argand) plane, and a complex number z therein.



The *conjugate* z^* of this complex number is defined to be

$$z^* = x - iy.$$

The *magnitude* (or *modulus,* or *absolute value*) $|z|$ of the complex number is defined to be the length ρ in Fig. 2.5, which per the Pythagorean theorem (§ 2.10) is such that

$$|z|^2 = x^2 + y^2. \tag{2.45}$$

The *phase* $\arg z$ of the complex number is defined to be the angle ϕ in the figure, which in terms of the trigonometric functions of § 3.1[26] is such that

$$\tan(\arg z) = \frac{y}{x}. \tag{2.46}$$

Specifically to extract the real and imaginary parts of a complex number, the notations

$$\begin{aligned} \Re(z) &= x, \\ \Im(z) &= y, \end{aligned} \tag{2.47}$$

are conventionally recognized (although often the symbols $\Re(\cdot)$ and $\Im(\cdot)$ are written $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$, particularly when printed by hand).

[26]This is a forward reference. If the equation doesn't make sense to you yet for this reason, skip it for now. The important point is that $\arg z$ is the angle ϕ in the figure.

2.12.1 Multiplication and division of complex numbers in rectangular form

Several elementary properties of complex numbers are readily seen if the fact that $i^2 = -1$ is kept in mind, including that

$$
\begin{aligned}
z_1 z_2 &= (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2), && (2.48)\\
\frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \left(\frac{x_2 - iy_2}{x_2 - iy_2}\right)\frac{x_1 + iy_1}{x_2 + iy_2} \\
&= \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}. && (2.49)
\end{aligned}
$$

It is a curious fact that

$$\frac{1}{i} = -i. \tag{2.50}$$

2.12.2 Complex conjugation

A very important property of complex numbers descends subtly from the fact that

$$i^2 = -1 = (-i)^2.$$

If one defined some number $j \equiv -i$, claiming that j not i were the true imaginary unit [10, § I:22-5], then one would find that

$$(-j)^2 = -1 = j^2,$$

and thus that all the basic properties of complex numbers in the j system held just as well as they did in the i system. The units i and j would differ indeed, but would perfectly mirror one another in every respect.

That is the basic idea. To establish it symbolically needs a page or so of slightly abstract algebra, the goal of which will be to show that $[f(z)]^* = f(z^*)$ for some unspecified function $f(z)$ with specified properties. To begin with, if

$$z = x + iy,$$

then

$$z^* = x - iy$$

by definition. Proposing that $(z^{k-1})^* = (z^*)^{k-1}$ (which may or may not be true but for the moment we assume it), we can write

$$
\begin{aligned}
z^{k-1} &= s_{k-1} + it_{k-1},\\
(z^*)^{k-1} &= s_{k-1} - it_{k-1},
\end{aligned}
$$

where s_{k-1} and t_{k-1} are symbols introduced to represent the real and imaginary parts of z^{k-1}. Multiplying the former equation by $z = x + iy$ and the latter by $z^* = x - iy$, we have that

$$\begin{aligned} z^k &= (xs_{k-1} - yt_{k-1}) + i(ys_{k-1} + xt_{k-1}), \\ (z^*)^k &= (xs_{k-1} - yt_{k-1}) - i(ys_{k-1} + xt_{k-1}). \end{aligned}$$

With the definitions $s_k \equiv xs_{k-1} - yt_{k-1}$ and $t_k \equiv ys_{k-1} + xt_{k-1}$, this is written more succinctly

$$\begin{aligned} z^k &= s_k + it_k, \\ (z^*)^k &= s_k - it_k. \end{aligned}$$

In other words, if $(z^{k-1})^* = (z^*)^{k-1}$, then it necessarily follows that $(z^k)^* = (z^*)^k$. The implication is reversible by reverse reasoning, so by mathematical induction[27] we have that

$$(z^k)^* = (z^*)^k \tag{2.51}$$

for all integral k. A direct consequence of this is that if

$$f(z) \equiv \sum_{k=-\infty}^{\infty} (a_k + ib_k)(z - z_o)^k, \tag{2.52}$$

$$f^*(z) \equiv \sum_{k=-\infty}^{\infty} (a_k - ib_k)(z - z_o^*)^k, \tag{2.53}$$

where a_k and b_k are real and imaginary parts of the coefficients peculiar to the function $f(\cdot)$, then

$$[f(z)]^* = f^*(z^*). \tag{2.54}$$

In the common case where all $b_k = 0$ and $z_o = x_o$ is a real number, then $f(\cdot)$ and $f^*(\cdot)$ are the same function, so (2.54) reduces to the desired form

$$[f(z)]^* = f(z^*), \tag{2.55}$$

which says that the effect of conjugating the function's input is merely to conjugate its output.

[27] *Mathematical induction* is an elegant old technique for the construction of mathematical proofs. Section 8.1 elaborates on the technique and offers a more extensive example. Beyond the present book, a very good introduction to mathematical induction is found in [14].

Equation (2.55) expresses a significant, general rule of complex numbers and complex variables which is better explained in words than in mathematical symbols. The rule is this: for most equations and systems of equations used to model physical systems, *one can produce an equally valid alternate model simply by simultaneously conjugating all the complex quantities present.*

(References: [14]; [24].)

2.12.3 Power series and analytic functions (preview)

Equation (2.52) is a general power series in $z - z_o$ [15, § 10.8]. Such power series have broad application.[28] It happens in practice that most functions of interest in modeling physical phenomena can conveniently be constructed as power series (or sums of power series)[29] with suitable choices of a_k, b_k and z_o.

The property (2.54) applies to all such functions, with (2.55) also applying to those for which $b_k = 0$ and $z_o = x_o$. The property the two equations represent is called the *conjugation property.* Basically, it says that if one replaces all the i in some mathematical model with $-i$, then the resulting conjugate model will be equally as valid as was the original.

Such functions, whether $b_k = 0$ and $z_o = x_o$ or not, are *analytic functions* (§ 8.4). In the formal mathematical definition, a function is analytic which is infinitely differentiable (Ch. 4) in the immediate domain neighborhood of interest. However, for applications a fair working definition of the analytic function might be "a function expressible as a power series." Ch. 8 elaborates. All power series are infinitely differentiable except at their poles.

There nevertheless exist one common group of functions which cannot be constructed as power series. These all have to do with the parts of complex numbers and have been introduced in this very section: the magnitude $|\cdot|$; the phase $\arg(\cdot)$; the conjugate $(\cdot)^*$; and the real and imaginary parts $\Re(\cdot)$ and $\Im(\cdot)$. These functions are not analytic and do not in general obey the

[28] That is a pretty impressive-sounding statement: "Such power series have broad application." However, air, words and molecules also have "broad application;" merely stating the fact does not tell us much. In fact the general power series is a sort of one-size-fits-all mathematical latex glove, which can be stretched to fit around almost any function. The interesting part is not so much in the general form (2.52) of the series as it is in the specific choice of a_k and b_k, which this section does not discuss.

Observe that the Taylor series (which this section also does not discuss; see § 8.3) is a power series with $a_k = b_k = 0$ for $k < 0$.

[29] The careful reader might observe that this statement neglects the *Gibbs' phenomenon,* but that curious matter will be dealt with in [section not yet written].

conjugation property. Also not analytic are the Heaviside unit step $u(t)$ and the Dirac delta $\delta(t)$ (§ 7.7), used to model discontinuities explicitly.

We shall have more to say about analytic functions in Ch. 8. We shall have more to say about complex numbers in §§ 3.11, 4.3.3, and 4.4, and much more yet in Chapter 5.

Chapter 3

Trigonometry

Trigonometry is the branch of mathematics which relates angles to lengths. This chapter introduces the trigonometric functions and derives their several properties.

3.1 Definitions

Consider the circle-inscribed right triangle of Fig. 3.1.

The angle ϕ in the diagram is given in *radians,* where a radian is the angle which, when centered in a unit circle, describes an arc of unit length.[1] Measured in radians, an angle ϕ intercepts an arc of curved length $\rho\phi$ on a circle of radius ρ. An angle in radians is a dimensionless number, so one need not write "$\phi = 2\pi/4$ radians;" it suffices to write "$\phi = 2\pi/4$." In math theory, we do angles in radians.

The angle of full revolution is given the symbol 2π—which thus is the circumference of a unit circle.[2] A quarter revolution, $2\pi/4$, is then the *right angle,* or square angle.

The trigonometric functions $\sin\phi$ and $\cos\phi$ (the "sine" and "cosine" of ϕ) relate the angle ϕ to the lengths shown in Fig. 3.1. The tangent function is then defined as

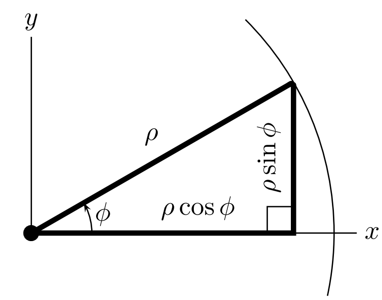
$$\tan\phi \equiv \frac{\sin\phi}{\cos\phi}, \tag{3.1}$$

which is the "rise" per unit "run," or *slope,* of the triangle's diagonal. In-

[1] The word "unit" means "one" in this context. A unit length is a length of 1 (not one centimeter or one mile, just an abstract 1). A unit circle is a circle of radius 1.

[2] Section 8.9 computes the numerical value of 2π.

Figure 3.1: The sine and the cosine (shown on a circle-inscribed right triangle, with the circle centered at the triangle's point).



verses of the three trigonometric functions can also be defined:

$$\begin{aligned} \arcsin(\sin\phi) &= \phi, \\ \arccos(\cos\phi) &= \phi, \\ \arctan(\tan\phi) &= \phi. \end{aligned}$$

When the last of these is written in the form

$$\arctan\left(\frac{y}{x}\right),$$

it is normally implied that x and y are to be interpreted as rectangular coordinates[3] and that the arctan function is to return ϕ in the correct quadrant $-\pi < \phi \le \pi$ (for example, $\arctan[1/(-1)] = [+3/8][2\pi]$, whereas $\arctan[(-1)/1] = [-1/8][2\pi]$). This is similarly the usual interpretation when an equation like

$$\tan\phi = \frac{y}{x}$$

is written.

By the Pythagorean theorem (§ 2.10), it is seen generally that[4]

$$\cos^2\phi + \sin^2\phi = 1. \tag{3.2}$$

[3] *Rectangular coordinates* are pairs of numbers (x, y) which uniquely specify points in a plane. Conventionally, the x coordinate indicates distance eastward; the y coordinate, northward. For instance, the coordinates $(3, -4)$ mean the point three units eastward and four units southward (that is, -4 units northward) from the *origin* $(0, 0)$. A third rectangular coordinate can also be added—(x, y, z)—where the z indicates distance upward.

[4] The notation $\cos^2\phi$ means $(\cos\phi)^2$.

Figure 3.2: The sine function.

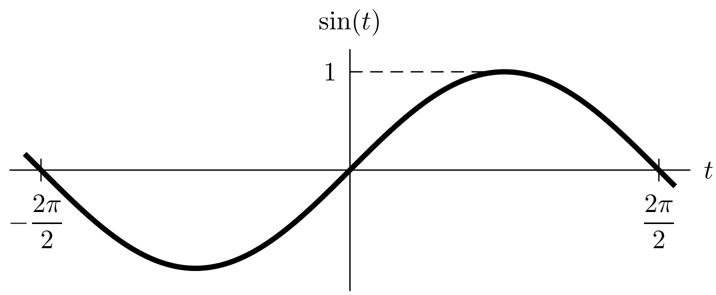


Fig. 3.2 plots the sine function. The shape in the plot is called a *sinusoid.*

3.2 Simple properties

Inspecting Fig. 3.1 and observing (3.1) and (3.2), one readily discovers the several simple trigonometric properties of Table 3.1.

3.3 Scalars, vectors, and vector notation

In applied mathematics, a *vector* is a magnitude of some kind coupled with a direction.[5] For example, "55 miles per hour northwestward" is a vector, as is the entity **u** depicted in Fig. 3.3. The entity **v** depicted in Fig. 3.4 is also a vector, in this case a three-dimensional one.

Many readers will already find the basic vector concept familiar, but for those who do not, a brief review: Vectors such as the

$$
\begin{aligned}
\mathbf{u} &= \hat{\mathbf{x}}x + \hat{\mathbf{y}}y, \\
\mathbf{v} &= \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z
\end{aligned}
$$

[5]The same word *vector* is also used to indicate an ordered set of N scalars (§ 8.10) or an $N \times 1$ matrix (Ch. 11); but those are not the uses of the word meant here.

Table 3.1: Simple properties of the trigonometric functions.

$$
\begin{aligned}
\sin(-\phi) &= -\sin\phi & \cos(-\phi) &= +\cos\phi \\
\sin(2\pi/4-\phi) &= +\cos\phi & \cos(2\pi/4-\phi) &= +\sin\phi \\
\sin(2\pi/2-\phi) &= +\sin\phi & \cos(2\pi/2-\phi) &= -\cos\phi \\
\sin(\phi\pm 2\pi/4) &= \pm\cos\phi & \cos(\phi\pm 2\pi/4) &= \mp\sin\phi \\
\sin(\phi\pm 2\pi/2) &= -\sin\phi & \cos(\phi\pm 2\pi/2) &= -\cos\phi \\
\sin(\phi+n2\pi) &= \sin\phi & \cos(\phi+n2\pi) &= \cos\phi
\end{aligned}
$$

$$
\begin{aligned}
\tan(-\phi) &= -\tan\phi \\
\tan(2\pi/4-\phi) &= +1/\tan\phi \\
\tan(2\pi/2-\phi) &= -\tan\phi \\
\tan(\phi\pm 2\pi/4) &= -1/\tan\phi \\
\tan(\phi\pm 2\pi/2) &= +\tan\phi \\
\tan(\phi+n2\pi) &= \tan\phi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin\phi}{\cos\phi} &= \tan\phi \\
\cos^2\phi+\sin^2\phi &= 1 \\
1+\tan^2\phi &= \frac{1}{\cos^2\phi} \\
1+\frac{1}{\tan^2\phi} &= \frac{1}{\sin^2\phi}
\end{aligned}
$$

Figure 3.3: A two-dimensional vector $\mathbf{u} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$, shown with its rectangular components.

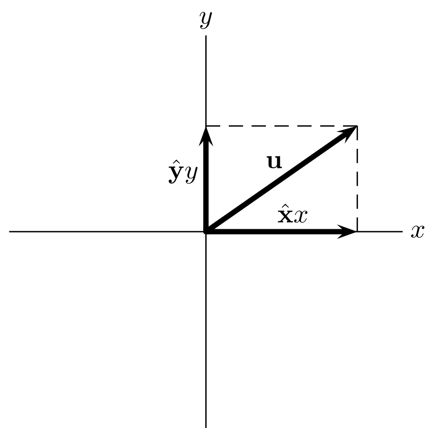
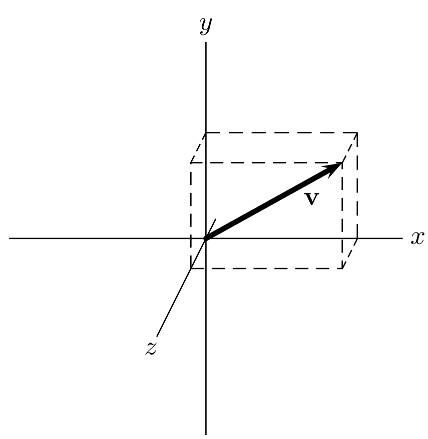


Figure 3.4: A three-dimensional vector $\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$.



of the figures are composed of multiples of the *unit basis vectors* $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, which themselves are vectors of unit length pointing in the cardinal directions their respective symbols suggest.[6] Any vector $\mathbf{a}$ can be factored into a *magnitude* a and a *unit vector* $\hat{\mathbf{a}}$, as

$$\mathbf{a} = \hat{\mathbf{a}}a,$$

where the $\hat{\mathbf{a}}$ represents direction only and has unit magnitude by definition, and where the a represents magnitude only and carries the physical units if any.[7] For example, $a = 55$ miles per hour, $\hat{\mathbf{a}} =$ northwestward. The unit vector $\hat{\mathbf{a}}$ itself can be expressed in terms of the unit basis vectors: for example, if $\hat{\mathbf{x}}$ points east and $\hat{\mathbf{y}}$ points north, then $\hat{\mathbf{a}} = -\hat{\mathbf{x}}(1/\sqrt{2})+\hat{\mathbf{y}}(1/\sqrt{2})$, where per the Pythagorean theorem $(-1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1^2$.

A single number which is not a vector or a matrix (Ch. 11) is called a *scalar.* In the example, $a = 55$ miles per hour is a scalar. Though the scalar a in the example happens to be real, scalars can be complex, too—which might surprise one, since scalars by definition lack direction and the Argand phase ϕ of Fig. 2.5 so strongly resembles a direction. However, phase is not an actual direction in the vector sense (the real number line in the Argand plane cannot be said to run west-to-east, or anything like that). The x, y and z of Fig. 3.4 are each (possibly complex) scalars; $\mathbf{v} =$

[6]Printing by hand, one customarily writes a general vector like $\mathbf{u}$ as "$\vec{u}$" or just "$\overline{u}$", and a unit vector like $\hat{\mathbf{x}}$ as "$\hat{x}$".

[7]The word "unit" here is unfortunately overloaded. As an adjective in mathematics, or in its noun form "unity," it refers to the number one (1)—not one mile per hour, one kilogram, one Japanese yen or anything like that; just an abstract 1. The word "unit" itself as a noun however usually signifies a physical or financial reference quantity of measure, like a mile per hour, a kilogram or even a Japanese yen. There is no inherent mathematical unity to 1 mile per hour (otherwise known as 0.447 meters per second, among other names). By contrast, a "unitless 1"—a 1 with no physical unit attached—does represent mathematical unity.

Consider the ratio $r = h_1/h_o$ of your height h_1 to my height h_o. Maybe you are taller than I am and so $r = 1.05$ (not 1.05 cm or 1.05 feet, just 1.05). Now consider the ratio h_1/h_1 of your height to your own height. That ratio is of course unity, exactly 1.

There is nothing ephemeral in the concept of mathematical unity, nor in the concept of unitless quantities in general. The concept is quite straightforward and entirely practical. That $r > 1$ means neither more nor less than that you are taller than I am. In applications, one often puts physical quantities in ratio precisely to strip the physical units from them, comparing the ratio to unity without regard to physical units.

$\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ is a vector. If x, y and z are complex, then[8]

$$\begin{aligned} |\mathbf{v}|^2 &= |x|^2 + |y|^2 + |z|^2 = x^*x + y^*y + z^*z \\ &= [\Re(x)]^2 + [\Im(x)]^2 + [\Re(y)]^2 + [\Im(y)]^2 \\ &\quad + [\Re(z)]^2 + [\Im(z)]^2 . \end{aligned} \tag{3.3}$$

A point is sometimes identified by the vector expressing its distance and direction from the origin of the coordinate system. That is, the point (x, y) can be identified with the vector $\hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. However, in the general case vectors are not associated with any particular origin; they represent distances and directions, not fixed positions.

Notice incidentally the relative orientation of the axes in Fig. 3.4. The axes are oriented such that if you point your flat right hand in the x direction, then bend your fingers in the y direction and extend your thumb, the thumb then points in the z direction. This is orientation by the *right-hand rule.* A left-handed orientation is equally possible, of course, but as neither orientation has a natural advantage over the other, we arbitrarily but conventionally accept the right-handed one as standard.[9]

3.4 Rotation

A fundamental problem in trigonometry arises when a vector

$$\mathbf{u} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y \tag{3.4}$$

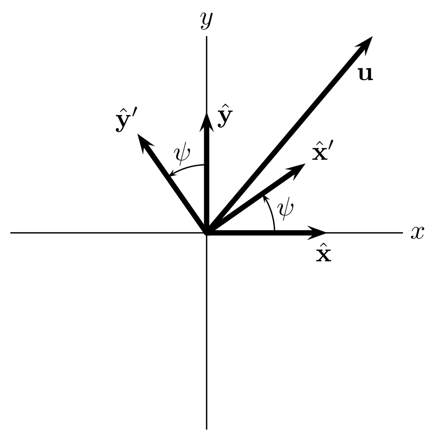
must be expressed in terms of alternate unit vectors $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$, where $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$ stand at right angles to one another and lie in the plane[10] of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, but are rotated from the latter by an angle ψ as depicted in Fig. 3.5.[11] In

[8] Some books print $|\mathbf{v}|$ as $\|\mathbf{v}\|$ to emphasize that it represents the real, scalar magnitude of a complex vector.

[9] The writer does not know the etymology for certain, but verbal lore in American engineering has it that the name "right-handed" comes from experience with a standard right-handed wood screw or machine screw. If you hold the screwdriver in your right hand and turn the screw in the natural manner clockwise, turning the screw slot from the x orientation toward the y, the screw advances away from you in the z direction into the wood or hole. If somehow you came across a left-handed screw, you'd probably find it easier to drive that screw with the screwdriver in your left hand.

[10] A *plane,* as the reader on this tier undoubtedly knows, is a flat (but not necessarily level) surface, infinite in extent unless otherwise specified. Space is three-dimensional. A plane is two-dimensional. A line is one-dimensional. A point is zero-dimensional. The

Figure 3.5: Vector basis rotation.



terms of the trigonometric functions of § 3.1, evidently

$$\begin{aligned} \hat{\mathbf{x}}' &= +\hat{\mathbf{x}}\cos\psi + \hat{\mathbf{y}}\sin\psi, \\ \hat{\mathbf{y}}' &= -\hat{\mathbf{x}}\sin\psi + \hat{\mathbf{y}}\cos\psi; \end{aligned} \tag{3.5}$$

and by appeal to symmetry it stands to reason that

$$\begin{aligned} \hat{\mathbf{x}} &= +\hat{\mathbf{x}}'\cos\psi - \hat{\mathbf{y}}'\sin\psi, \\ \hat{\mathbf{y}} &= +\hat{\mathbf{x}}'\sin\psi + \hat{\mathbf{y}}'\cos\psi. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.4) yields

$$\mathbf{u} = \hat{\mathbf{x}}'(x\cos\psi + y\sin\psi) + \hat{\mathbf{y}}'(-x\sin\psi + y\cos\psi), \tag{3.7}$$

which was to be derived.

Equation (3.7) finds general application where rotations in rectangular coordinates are involved. If the question is asked, "what happens if I rotate not the unit basis vectors but rather the vector $\mathbf{u}$ instead?" the answer is

plane belongs to this geometrical hierarchy.

[11]The "$'$" mark is pronounced "prime" or "primed" (for no especially good reason of which the author is aware, but anyway, that's how it's pronounced). Mathematical writing employs the mark for a variety of purposes. Here, the mark merely distinguishes the new unit vector $\hat{\mathbf{x}}'$ from the old $\hat{\mathbf{x}}$.

that it amounts to the same thing, except that the sense of the rotation is reversed:

$$\mathbf{u}' = \hat{\mathbf{x}}(x\cos\psi - y\sin\psi) + \hat{\mathbf{y}}(x\sin\psi + y\cos\psi). \tag{3.8}$$

Whether it is the basis or the vector which rotates thus depends on your point of view.[12]

3.5 Trigonometric functions of sums and differences of angles

With the results of § 3.4 in hand, we now stand in a position to consider trigonometric functions of sums and differences of angles. Let

$$\begin{aligned}\hat{\mathbf{a}} &\equiv \hat{\mathbf{x}}\cos\alpha + \hat{\mathbf{y}}\sin\alpha,\\ \hat{\mathbf{b}} &\equiv \hat{\mathbf{x}}\cos\beta + \hat{\mathbf{y}}\sin\beta,\end{aligned}$$

be vectors of unit length in the xy plane, respectively at angles α and β from the x axis. If we wanted $\hat{\mathbf{b}}$ to coincide with $\hat{\mathbf{a}}$, we would have to rotate it by $\psi = \alpha - \beta$. According to (3.8) and the definition of $\hat{\mathbf{b}}$, if we did this we would obtain

$$\begin{aligned}\hat{\mathbf{b}}' = \hat{\mathbf{x}}&[\cos\beta\cos(\alpha-\beta) - \sin\beta\sin(\alpha-\beta)]\\ &+ \hat{\mathbf{y}}[\cos\beta\sin(\alpha-\beta) + \sin\beta\cos(\alpha-\beta)].\end{aligned}$$

Since we have deliberately chosen the angle of rotation such that $\hat{\mathbf{b}}' = \hat{\mathbf{a}}$, we can separately equate the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms in the expressions for $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}'$ to obtain the pair of equations

$$\begin{aligned}\cos\alpha &= \cos\beta\cos(\alpha-\beta) - \sin\beta\sin(\alpha-\beta),\\ \sin\alpha &= \cos\beta\sin(\alpha-\beta) + \sin\beta\cos(\alpha-\beta).\end{aligned}$$

[12]This is only true, of course, with respect to the vectors themselves. When one actually rotates a physical body, the body experiences forces during rotation which might or might not change the body internally in some relevant way.

Solving the last pair simultaneously[13] for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$ and observing that $\sin^2(\cdot) + \cos^2(\cdot) = 1$ yields

$$\begin{aligned} \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta, \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta. \end{aligned} \tag{3.9}$$

With the change of variable $\beta \leftarrow -\beta$ and the observations from Table 3.1 that $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = +\cos(\phi)$, eqns. (3.9) become

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned} \tag{3.10}$$

Equations (3.9) and (3.10) are the basic formulas for trigonometric functions of sums and differences of angles.

3.5.1 Variations on the sums and differences

Several useful variations on (3.9) and (3.10) are achieved by combining the equations in various straightforward ways.[14] These include

$$\begin{aligned} \sin\alpha\sin\beta &= \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}, \\ \sin\alpha\cos\beta &= \frac{\sin(\alpha - \beta) + \sin(\alpha + \beta)}{2}, \\ \cos\alpha\cos\beta &= \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}. \end{aligned} \tag{3.11}$$

[13]The easy way to do this is

- to subtract $\sin\beta$ times the first equation from $\cos\beta$ times the second, then to solve the result for $\sin(\alpha - \beta)$;
- to add $\cos\beta$ times the first equation to $\sin\beta$ times the second, then to solve the result for $\cos(\alpha - \beta)$.

This shortcut technique for solving a pair of equations simultaneously for a pair of variables is well worth mastering. In this book alone, it proves useful many times.

[14]Refer to footnote 13 above for the technique.

With the change of variables $\delta \leftarrow \alpha - \beta$ and $\gamma \leftarrow \alpha + \beta$, (3.9) and (3.10) become

$$\begin{aligned}
\sin\delta &= \sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) - \cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\
\cos\delta &= \cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) + \sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\
\sin\gamma &= \sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) + \cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\
\cos\gamma &= \cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) - \sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right).
\end{aligned}$$

Combining these in various ways, we have that

$$\begin{aligned}
\sin\gamma + \sin\delta &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right), \\
\sin\gamma - \sin\delta &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\
\cos\delta + \cos\gamma &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right), \\
\cos\delta - \cos\gamma &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right).
\end{aligned} \tag{3.12}$$

3.5.2 Trigonometric functions of double and half angles

If $\alpha = \beta$, then eqns. (3.10) become the *double-angle formulas*

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha, \\
\cos 2\alpha &= 2\cos^2\alpha - 1 = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha.
\end{aligned} \tag{3.13}$$

Solving (3.13) for $\sin^2\alpha$ and $\cos^2\alpha$ yields the *half-angle formulas*

$$\begin{aligned}
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2}, \\
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2}.
\end{aligned} \tag{3.14}$$

3.6 Trigonometric functions of the hour angles

In general one uses the Taylor series of Ch. 8 to calculate trigonometric functions of specific angles. However, for angles which happen to be integral multiples of an *hour*—there are twenty-four or 0x18 hours in a circle, just as there are twenty-four or 0x18 hours in a day[15]—for such angles simpler expressions exist. Figure 3.6 shows the angles. Since such angles arise very frequently in practice, it seems worth our while to study them specially.

Table 3.2 tabulates the trigonometric functions of these *hour angles.* To see how the values in the table are calculated, look at the square and the equilateral triangle[16] of Fig. 3.7. Each of the square's four angles naturally measures six hours; and since a triangle's angles always total twelve hours (§ 2.9.3), by symmetry each of the angles of the equilateral triangle in the figure measures four. Also by symmetry, the perpendicular splits the triangle's top angle into equal halves of two hours each and its bottom leg into equal segments of length 1/2 each; and the diagonal splits the square's corner into equal halves of three hours each. The Pythagorean theorem (§ 2.10) then supplies the various other lengths in the figure, after which we observe from Fig. 3.1 that

- the sine of a non-right angle in a right triangle is the opposite leg's length divided by the diagonal's,

[15]Hence an hour is 15°, but you weren't going to write your angles in such inelegant conventional notation as "15°," were you? Well, if you were, you're in good company.

The author is fully aware of the barrier the unfamiliar notation poses for most first-time readers of the book. The barrier is erected neither lightly nor disrespectfully. Consider:

- There are 0x18 hours in a circle.
- There are 360 degrees in a circle.

Both sentences say the same thing, don't they? But even though the "0x" hex prefix is a bit clumsy, the first sentence nevertheless says the thing rather better. The reader is urged to invest the attention and effort to master the notation.

There is a psychological trap regarding the hour. The familiar, standard clock face shows only twelve hours not twenty-four, so the angle between eleven o'clock and twelve *on the clock face* is not an hour of arc! That angle is two hours of arc. This is because the clock face's geometry is artificial. If you have ever been to the Old Royal Observatory at Greenwich, England, you may have seen the big clock face there with all twenty-four hours on it. It'd be a bit hard to read the time from such a crowded clock face were it not so big, but anyway, the angle between hours on the Greenwich clock is indeed an honest hour of arc. [4]

The hex and hour notations are recommended mostly only for theoretical math work. It is not claimed that they offer much benefit in most technical work of the less theoretical kinds. If you wrote an engineering memo describing a survey angle as 0x1.80 hours instead of 22.5 degrees, for example, you'd probably not like the reception the memo got. Nonetheless, the improved notation fits a book of this kind so well that the author hazards it. It is hoped that after trying the notation a while, the reader will approve the choice.

[16]An *equilateral* triangle is, as the name and the figure suggest, a triangle whose three sides all have the same length.

Figure 3.6: The 0x18 hours in a circle.

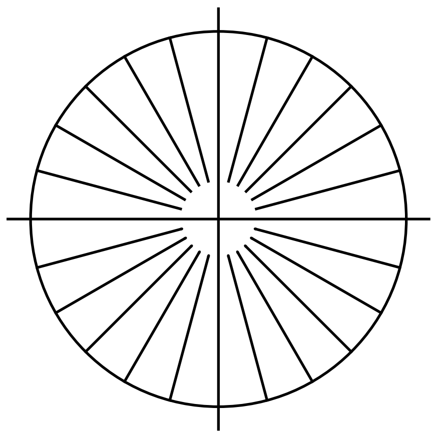


Figure 3.7: A square and an equilateral triangle for calculating trigonometric functions of the hour angles.

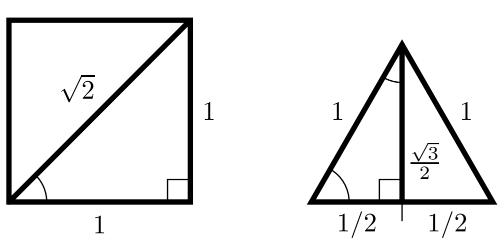
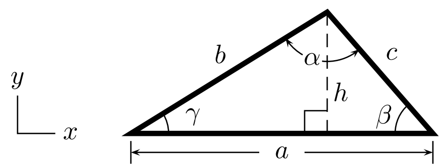


Table 3.2: Trigonometric functions of the hour angles.

ANGLE ϕ [radians]	[hours]	$\sin\phi$	$\tan\phi$	$\cos\phi$
0	0	0	0	1
$\frac{2\pi}{\mathrm{0x18}}$	1	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
$\frac{2\pi}{\mathrm{0xC}}$	2	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{3}}{2}$
$\frac{2\pi}{8}$	3	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$
$\frac{2\pi}{6}$	4	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{1}{2}$
$\frac{(5)(2\pi)}{\mathrm{0x18}}$	5	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
$\frac{2\pi}{4}$	6	1	∞	0

Figure 3.8: The laws of sines and cosines.



- the tangent is the opposite leg's length divided by the adjacent leg's, and

- the cosine is the adjacent leg's length divided by the diagonal's.

With this observation and the lengths in the figure, one can calculate the sine, tangent and cosine of angles of two, three and four hours.

The values for one and five hours are found by applying (3.9) and (3.10) against the values for two and three hours just calculated. The values for zero and six hours are, of course, seen by inspection.[17]

3.7 The laws of sines and cosines

Refer to the triangle of Fig. 3.8. By the definition of the sine function, one can write that

$$c \sin \beta = h = b \sin \gamma,$$

or in other words that

$$\frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

[17]The creative reader may notice that he can extend the table to any angle by repeated application of the various sum, difference and half-angle formulas from the preceding sections to the values already in the table. However, the Taylor series (§ 8.7) offers a cleaner, quicker way to calculate trigonometrics of non-hour angles.

But there is nothing special about β and γ; what is true for them must be true for α, too.[18] Hence,

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}. \tag{3.15}$$

This equation is known as *the law of sines.*

On the other hand, if one expresses a and b as vectors emanating from the point γ,[19]

$$\begin{aligned}\mathbf{a} &= \hat{\mathbf{x}}a,\\ \mathbf{b} &= \hat{\mathbf{x}}b\cos\gamma + \hat{\mathbf{y}}b\sin\gamma,\end{aligned}$$

then

$$\begin{aligned}c^2 &= |\mathbf{b}-\mathbf{a}|^2\\ &= (b\cos\gamma - a)^2 + (b\sin\gamma)^2\\ &= a^2 + (b^2)(\cos^2\gamma + \sin^2\gamma) - 2ab\cos\gamma.\end{aligned}$$

Since $\cos^2(\cdot) + \sin^2(\cdot) = 1$, this is

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \tag{3.16}$$

This equation is known as *the law of cosines.*

3.8 Summary of properties

Table 3.2 on page 56 has listed the values of trigonometric functions of the hour angles. Table 3.1 on page 46 has summarized simple properties of the trigonometric functions. Table 3.3 summarizes further properties, gathering them from §§ 3.4, 3.5 and 3.7.

[18] "But," it is objected, "there *is* something special about α. The perpendicular h drops from it."

True. However, the h is just a utility variable to help us to manipulate the equation into the desired form; we're not interested in h itself. Nothing prevents us from dropping additional perpendiculars h_β and h_γ from the other two corners and using those as utility variables, too, if we like. We can use any utility variables we want.

[19] Here is another example of the book's judicious relaxation of formal rigor. Of course there is no "point γ;" γ is an angle not a point. However, the writer suspects in light of Fig. 3.8 that few readers will be confused as to which point is meant. The skillful applied mathematician does not multiply labels without need.

Table 3.3: Further properties of the trigonometric functions.

$$
\begin{aligned}
\mathbf{u} &= \hat{\mathbf{x}}'(x\cos\psi + y\sin\psi) + \hat{\mathbf{y}}'(-x\sin\psi + y\cos\psi) \\
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin\alpha\sin\beta &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \\
\sin\alpha\cos\beta &= \frac{\sin(\alpha-\beta) + \sin(\alpha+\beta)}{2} \\
\cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2} \\
\sin\gamma + \sin\delta &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) \\
\sin\gamma - \sin\delta &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right) \\
\cos\delta + \cos\gamma &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) \\
\cos\delta - \cos\gamma &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right) \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= 2\cos^2\alpha - 1 = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha \\
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2} \\
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2} \\
\frac{\sin\gamma}{c} &= \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \\
c^2 &= a^2 + b^2 - 2ab\cos\gamma
\end{aligned}
$$

3.9 Cylindrical and spherical coordinates

Section 3.3 has introduced the concept of the vector

$$\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z.$$

The coefficients (x, y, z) on the equation's right side are *coordinates*—specifically, *rectangular coordinates*—which given a specific orthonormal[20] set of unit basis vectors $[\hat{\mathbf{x}}\ \hat{\mathbf{y}}\ \hat{\mathbf{z}}]$ uniquely identify a point (see Fig. 3.4 on page 47). Such rectangular coordinates are simple and general, and are convenient for many purposes. However, there are at least two broad classes of conceptually simple problems for which rectangular coordinates tend to be inconvenient: problems in which an axis or a point dominates. Consider for example an electric wire's magnetic field, whose intensity varies with distance from the wire (an axis); or the illumination a lamp sheds on a printed page of this book, which depends on the book's distance from the lamp (a point).

To attack a problem dominated by an axis, the *cylindrical coordinates* $(\rho; \phi, z)$ can be used instead of the rectangular coordinates (x, y, z). To attack a problem dominated by a point, the *spherical coordinates* $(r; \theta; \phi)$ can be used.[21] Refer to Fig. 3.9. Such coordinates are related to one another and to the rectangular coordinates by the formulas of Table 3.4.

Cylindrical and spherical coordinates can greatly simplify the analyses of the kinds of problems they respectively fit, but they come at a price. There are no constant unit basis vectors to match them. That is,

$$\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z \neq \hat{\boldsymbol{\rho}}\rho + \hat{\boldsymbol{\phi}}\phi + \hat{\mathbf{z}}z \neq \hat{\mathbf{r}}r + \hat{\boldsymbol{\theta}}\theta + \hat{\boldsymbol{\phi}}\phi.$$

It doesn't work that way. Nevertheless *variable* unit basis vectors are defined:

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &\equiv +\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \\
\hat{\boldsymbol{\phi}} &\equiv -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi, \\
\hat{\mathbf{r}} &\equiv +\hat{\mathbf{z}}\cos\theta + \hat{\boldsymbol{\rho}}\sin\theta, \\
\hat{\boldsymbol{\theta}} &\equiv -\hat{\mathbf{z}}\sin\theta + \hat{\boldsymbol{\rho}}\cos\theta.
\end{aligned} \tag{3.17}$$

Such variable unit basis vectors point locally in the directions in which their respective coordinates advance.

[20] *Orthonormal* in this context means "of unit length and at right angles to the other vectors in the set." [3, "Orthonormality," 14:19, 7 May 2006]

[21] Notice that the ϕ is conventionally written second in cylindrical $(\rho; \phi, z)$ but third in spherical $(r; \theta; \phi)$ coordinates. This odd-seeming convention is to maintain proper right-handed coordinate rotation. It will be clearer after reading Ch. [not yet written].

Figure 3.9: A point on a sphere, in spherical $(r; \theta; \phi)$ and cylindrical $(\rho; \phi, z)$ coordinates. (The axis labels bear circumflexes in this figure only to disambiguate the $\hat{z}$ axis from the cylindrical coordinate z.)

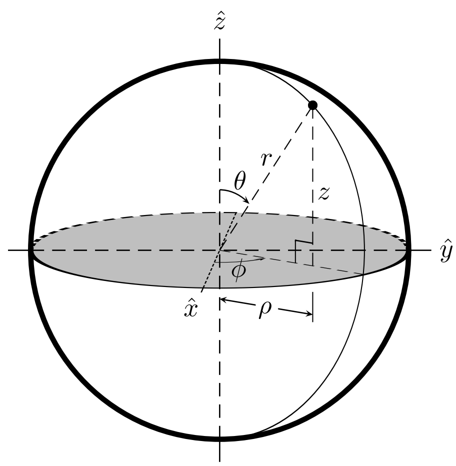


Table 3.4: Relations among the rectangular, cylindrical and spherical coordinates.

$$
\begin{aligned}
\rho^2 &= x^2 + y^2 \\
r^2 &= \rho^2 + z^2 = x^2 + y^2 + z^2 \\
\tan\theta &= \frac{\rho}{z} \\
\tan\phi &= \frac{y}{x} \\
z &= r\cos\theta \\
\rho &= r\sin\theta \\
x &= \rho\cos\phi = r\sin\theta\cos\phi \\
y &= \rho\sin\phi = r\sin\theta\sin\phi
\end{aligned}
$$

Convention usually orients $\hat{\mathbf{z}}$ in the direction of a problem's axis. Occasionally however a problem arises in which it is more convenient to orient $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ in the direction of the problem's axis (usually because $\hat{\mathbf{z}}$ has already been established in the direction of some other pertinent axis). Changing the meanings of known symbols like ρ, θ and ϕ is usually not a good idea, but you can use symbols like

$$\begin{aligned} \rho_x^2 &= y^2 + z^2, & \rho_y^2 &= z^2 + x^2, \\ \tan\theta_x &= \frac{\rho_x}{x}, & \tan\theta_y &= \frac{\rho_y}{y}, \\ \tan\phi_x &= \frac{z}{y}, & \tan\phi_y &= \frac{x}{z}, \end{aligned} \tag{3.18}$$

instead if needed.[22]

3.10 The complex triangle inequalities

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ represent the three sides of a triangle such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, then per (2.39)

$$|\mathbf{a}| - |\mathbf{b}| \le |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|.$$

These are just the triangle inequalities of § 2.9.2 in vector notation. But if the triangle inequalities hold for vectors in a plane, then why not equally for complex numbers? Consider the geometric interpretation of the Argand plane of Fig. 2.5 on page 37. Evidently,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2| \tag{3.19}$$

for any two complex numbers z_1 and z_2. Extending the sum inequality, we have that

$$\left|\sum_k z_k\right| \le \sum_k |z_k|. \tag{3.20}$$

Naturally, (3.19) and (3.20) hold equally well for real numbers as for complex; one may find the latter formula useful for sums of real numbers, for example, when some of the numbers summed are positive and others negative.

[22]Symbols like ρ_x are logical but, as far as this writer is aware, not standard. The writer is not aware of any conventionally established symbols for quantities like these.

An important consequence of (3.20) is that if $\sum |z_k|$ converges, then $\sum z_k$ also converges. Such a consequence is important because mathematical derivations sometimes need the convergence of $\sum z_k$ established, which can be hard to do directly. Convergence of $\sum |z_k|$, which per (3.20) implies convergence of $\sum z_k$, is often easier to establish.

3.11 De Moivre's theorem

Compare the Argand-plotted complex number of Fig. 2.5 (page 37) against the vector of Fig. 3.3 (page 47). Although complex numbers are scalars not vectors, the figures do suggest an analogy between complex phase and vector direction. With reference to Fig. 2.5 we can write

$$z = (\rho)(\cos\phi + i\sin\phi) = \rho \operatorname{cis}\phi, \tag{3.21}$$

where

$$\operatorname{cis}\phi \equiv \cos\phi + i\sin\phi. \tag{3.22}$$

If $z = x + iy$, then evidently

$$\begin{aligned} x &= \rho\cos\phi, \\ y &= \rho\sin\phi. \end{aligned} \tag{3.23}$$

Per (2.48),

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2).$$

Applying (3.23) to the equation yields

$$\frac{z_1 z_2}{\rho_1 \rho_2} = (\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2) + i(\sin\phi_1 \cos\phi_2 + \cos\phi_1 \sin\phi_2).$$

But according to (3.10), this is just

$$\frac{z_1 z_2}{\rho_1 \rho_2} = \cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2),$$

or in other words

$$z_1 z_2 = \rho_1 \rho_2 \operatorname{cis}(\phi_1 + \phi_2). \tag{3.24}$$

Equation (3.24) is an important result. It says that if you want to multiply complex numbers, it suffices

- to multiply their magnitudes and

- to add their phases.

It follows by parallel reasoning (or by extension) that

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \operatorname{cis}(\phi_1 - \phi_2) \tag{3.25}$$

and by extension that

$$z^a = \rho^a \operatorname{cis} a\phi. \tag{3.26}$$

Equations (3.24), (3.25) and (3.26) are known as *de Moivre's theorem.*[23] [24][3]

We have not shown it yet, but in § 5.4 we shall show that

$$\operatorname{cis} \phi = \exp i\phi = e^{i\phi},$$

where $\exp(\cdot)$ is the natural exponential function and e is the natural logarithmic base, both defined in Ch. 5. De Moivre's theorem is most useful in light of this.

[23] Also called *de Moivre's formula.* Some authors apply the name of de Moivre directly only to (3.26), or to some variation thereof; but as the three equations express essentially the same idea, if you refer to any of them as *de Moivre's theorem* then you are unlikely to be misunderstood.

Chapter 4

The derivative

The mathematics of *calculus* concerns a complementary pair of questions:[1]

- Given some function $f(t)$, what is the function's instantaneous rate of change, or *derivative,* $f'(t)$?
- Interpreting some function $f'(t)$ as an instantaneous rate of change, what is the corresponding accretion, or *integral,* $f(t)$?

This chapter builds toward a basic understanding of the first question.

4.1 Infinitesimals and limits

Calculus systematically treats numbers so large and so small, they lie beyond the reach of our mundane number system.

[1]Although once grasped the concept is relatively simple, to understand this pair of questions, so briefly stated, is no trivial thing. They are the pair which eluded or confounded the most brilliant mathematical minds of the ancient world.

The greatest conceptual hurdle—the stroke of brilliance—probably lies simply in stating the pair of questions clearly. Sir Isaac Newton and G.W. Leibnitz cleared this hurdle for us in the seventeenth century, so now at least we know the right pair of questions to ask. With the pair in hand, the calculus beginner's first task is quantitatively to understand the pair's interrelationship, generality and significance. Such an understanding constitutes the basic calculus concept.

It cannot be the role of a book like this one to lead the beginner gently toward an apprehension of the basic calculus concept. Once grasped, the concept is simple and briefly stated. In this book we necessarily state the concept briefly, then move along. Many instructional textbooks—[14] is a worthy example—have been written to lead the beginner gently. Although a sufficiently talented, dedicated beginner could perhaps obtain the basic calculus concept directly here, he would probably find it quicker and more pleasant to begin with a book like the one referenced.

4.1.1 The infinitesimal

A number ϵ is an *infinitesimal* if it is so small that

$$0 < |\epsilon| < a$$

for all possible mundane positive numbers a.

This is somewhat a difficult concept, so if it is not immediately clear then let us approach the matter colloquially. Let me propose to you that I have an infinitesimal.

"How big is your infinitesimal?" you ask.

"Very, very small," I reply.

"How small?"

"Very small."

"Smaller than 0x0.01?"

"Smaller than what?"

"Than 2^{-8}. You said that we should use hexadecimal notation in this book, remember?"

"Sorry. Yes, right, smaller than 0x0.01."

"What about 0x0.0001? Is it smaller than that?"

"Much smaller."

"Smaller than 0x0.0000 0000 0000 0001?"

"Smaller."

"Smaller than $2^{-\mathrm{0x1\,0000\,0000\,0000\,0000}}$?"

"Now *that* is an impressively small number. Nevertheless, my infinitesimal is smaller still."

"Zero, then."

"Oh, no. Bigger than that. My infinitesimal is definitely bigger than zero."

This is the idea of the infinitesimal. It is a definite number of a certain nonzero magnitude, but its smallness conceptually lies beyond the reach of our mundane number system.

If ϵ is an infinitesimal, then $1/\epsilon$ can be regarded as an *infinity:* a very large number much larger than any mundane number one can name.

The principal advantage of using symbols like ϵ rather than 0 for infinitesimals is in that it permits us conveniently to compare one infinitesimal against another, to add them together, to divide them, etc. For instance, if $\delta = 3\epsilon$ is another infinitesimal, then the quotient δ/ϵ is not some unfathomable 0/0; rather it is $\delta/\epsilon = 3$. In physical applications, the infinitesimals are often not true mathematical infinitesimals but rather relatively very small quantities such as the mass of a wood screw compared to the mass of a wooden house frame, or the audio power of your voice compared to that of a jet engine. The additional cost of inviting one more guest to the wedding may or may not be infinitesimal, depending on your point of view. The key point is that the infinitesimal quantity be negligible by comparison, whatever "negligible" means in the context.[2]

The second-order infinitesimal ϵ^2 is so small on the scale of the common, first-order infinitesimal ϵ that the even latter cannot measure it. The ϵ^2 is an infinitesimal to the infinitesimals. Third- and higher-order infinitesimals are likewise possible.

The notation $u \ll v$, or $v \gg u$, indicates that u is much less than v, typically such that one can regard the quantity u/v to be an infinitesimal. In fact, one common way to specify that ϵ be infinitesimal is to write that $\epsilon \ll 1$.

4.1.2 Limits

The notation $\lim_{z \to z_o}$ indicates that z draws as near to z_o as it possibly can. When written $\lim_{z \to z_o^+}$, the implication is that z draws toward z_o from the positive side such that $z > z_o$. Similarly, when written $\lim_{z \to z_o^-}$, the implication is that z draws toward z_o from the negative side.

The reason for the notation is to provide a way to handle expressions like

$$\frac{3z}{2z}$$

as z vanishes:

$$\lim_{z \to 0} \frac{3z}{2z} = \frac{3}{2}.$$

The symbol "$\lim_Q$" is short for "in the limit as Q."

Notice that lim is not a function like log or sin. It is just a reminder that a quantity approaches some value, used when saying that the quantity

[2]Among scientists and engineers who study wave phenomena, there is an old rule of thumb that sinusoidal waveforms be discretized not less finely than ten points per wavelength. In keeping with this book's adecimal theme (Appendix A) and the concept of the hour of arc (§ 3.6), we should probably render the rule as *twelve* points per wavelength here. In any case, even very roughly speaking, a quantity greater then 1/0xC of the principal to which it compares probably cannot rightly be regarded as infinitesimal. On the other hand, a quantity less than 1/0x10000 of the principal is indeed infinitesimal for most practical purposes (but not all: for example, positions of spacecraft and concentrations of chemical impurities must sometimes be accounted more precisely). For quantities between 1/0xC and 1/0x10000, it depends on the accuracy one seeks.

equaled the value would be confusing. Consider that to say

$$\lim_{z\to 2^-}(z+2) = 4$$

is just a fancy way of saying that $2+2=4$. The lim notation is convenient to use sometimes, but it is not magical. Don't let it confuse you.

4.2 Combinatorics

In its general form, the problem of selecting k specific items out of a set of n available items belongs to probability theory ([chapter not yet written]). In its basic form, however, the same problem also applies to the handling of polynomials or power series. This section treats the problem in its basic form. [14]

4.2.1 Combinations and permutations

Consider the following scenario. I have several small wooden blocks of various shapes and sizes, painted different colors so that you can clearly tell each block from the others. If I offer you the blocks and you are free to take all, some or none of them at your option, if you can take whichever blocks you want, then how many distinct choices of blocks do you have? Answer: you have 2^n choices, because you can accept or reject the first block, then accept or reject the second, then the third, and so on.

Now, suppose that what you want is exactly k blocks, neither more nor fewer. Desiring exactly k blocks, you select your favorite block first: there are n options for this. Then you select your second favorite: for this, there are $n-1$ options (why not n options? because you have already taken one block from me; I have only $n-1$ blocks left). Then you select your third favorite—for this there are $n-2$ options—and so on until you have k blocks. There are evidently

$$P\begin{pmatrix} n \\ k \end{pmatrix} \equiv n!/(n-k)! \tag{4.1}$$

ordered ways, or *permutations,* available for you to select exactly k blocks.

However, some of these distinct permutations result in putting exactly the same *combination* of blocks in your pocket; for instance, the permutations red-green-blue and green-red-blue constitute the same combination, whereas red-white-blue is a different combination entirely. For a single combination of k blocks (red, green, blue), evidently $k!$ permutations are possible (red-green-blue, red-blue-green, green-red-blue, green-blue-red, blue-red-green, blue-green-red). Hence dividing the number of permutations (4.1) by $k!$ yields the number of combinations

$$\binom{n}{k} \equiv \frac{n!/(n-k)!}{k!}. \tag{4.2}$$

Properties of the number $\binom{n}{k}$ of combinations include that

$$\begin{aligned}
\binom{n}{n-k} &= \binom{n}{k}, &(4.3)\\
\sum_{k=0}^{n} \binom{n}{k} &= 2^n, &(4.4)\\
\binom{n-1}{k-1} + \binom{n-1}{k} &= \binom{n}{k}, &(4.5)\\
\binom{n}{k} &= \frac{n-k+1}{k}\binom{n}{k-1} &(4.6)\\
&= \frac{k+1}{n-k}\binom{n}{k+1} &(4.7)\\
&= \frac{n}{k}\binom{n-1}{k-1} &(4.8)\\
&= \frac{n}{n-k}\binom{n-1}{k}. &(4.9)
\end{aligned}$$

Equation (4.3) results from changing the variable $k \leftarrow n - k$ in (4.2). Equation (4.4) comes directly from the observation (made at the head of this section) that 2^n total combinations are possible if any k is allowed. Equation (4.5) is seen when an nth block—let us say that it is a black block—is added to an existing set of $n-1$ blocks; to choose k blocks then, you can either choose k from the original set, or the black block plus $k-1$ from the original set. Equations (4.6) through (4.9) come directly from the definition (4.2); they relate combinatoric coefficients to their neighbors in Pascal's triangle (§ 4.2.2).

Because one cannot choose fewer than zero or more than n from n blocks,

$$\binom{n}{k} = 0 \quad \text{unless } 0 \le k \le n. \tag{4.10}$$

For $\binom{n}{k}$ when $n < 0$, there is no obvious definition.

Figure 4.1: The plan for Pascal's triangle.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\binom{1}{1} \\
\binom{2}{0}\binom{2}{1}\binom{2}{2} \\
\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3} \\
\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4} \\
\vdots
\end{array}
$$

4.2.2 Pascal's triangle

Consider the triangular layout in Fig. 4.1 of the various possible $\binom{n}{k}$. Evaluated, this yields Fig. 4.2, *Pascal's triangle.* Notice how each entry in the triangle is the sum of the two entries immediately above, as (4.5) predicts. (In fact this is the easy way to fill Pascal's triangle out: for each entry, just add the two entries above.)

4.3 The binomial theorem

This section presents the binomial theorem and one of its significant consequences.

4.3.1 Expanding the binomial

The *binomial theorem* holds that[3]

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \tag{4.11}$$

[3]The author is given to understand that, by an heroic derivational effort, (4.11) can be extended to nonintegral n. However, since applied mathematics does not usually concern itself with hard theorems of little known practical use, the extension is not covered in this book.

Figure 4.2: Pascal's triangle.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad \mathrm{A} \quad \mathrm{A} \quad 5 \quad 1 \\
1 \quad 6 \quad \mathrm{F} \quad 14 \quad \mathrm{F} \quad 6 \quad 1 \\
1 \quad 7 \quad 15 \; 23 \; 23 \; 15 \quad 7 \quad 1 \\
\vdots
\end{array}
$$

In the common case $a = 1$, $b = \epsilon \ll 1$, this is

$$(1+\epsilon)^n = \sum_{k=0}^{n} \binom{n}{k} \epsilon^k \tag{4.12}$$

(actually this holds for any ϵ, small or large; but the typical case of interest is $\epsilon \ll 1$). In either form, the binomial theorem is a direct consequence of the combinatorics of § 4.2. Since

$$(a+b)^n = (a+b)(a+b)\cdots(a+b)(a+b),$$

each $(a+b)$ factor corresponds to one of the "wooden blocks," where a means rejecting the block and b, accepting it.

4.3.2 Powers of numbers near unity

Since $\binom{n}{0} = 1$ and $\binom{n}{1} = n$, it follows from (4.12) that

$$(1+\epsilon_o)^n \approx 1 + n\epsilon_o,$$

to excellent precision for nonnegative integral n if ϵ_o is sufficiently small. Furthermore, raising the equation to the $1/n$ power then changing $\delta \leftarrow n\epsilon_o$ and $m \leftarrow n$, we have that

$$(1+\delta)^{1/m} \approx 1 + \frac{\delta}{m}.$$

Changing $1+\delta \leftarrow (1+\epsilon)^n$ and observing from the $(1+\epsilon_o)^n$ equation above that this implies that $\delta \approx n\epsilon$, we have that

$$(1+\epsilon)^{n/m} \approx 1 + \frac{n}{m}\epsilon.$$

Inverting this equation yields

$$(1+\epsilon)^{-n/m} \approx \frac{1}{1+(n/m)\epsilon} = \frac{[1-(n/m)\epsilon]}{[1-(n/m)\epsilon][1+(n/m)\epsilon]} \approx 1 - \frac{n}{m}\epsilon.$$

Taken together, the last two equations imply that

$$(1+\epsilon)^x \approx 1 + x\epsilon \tag{4.13}$$

for any real x.

The writer knows of no conventional name[4] for (4.13), but named or unnamed it is an important equation. The equation offers a simple, accurate way of approximating any real power of numbers in the near neighborhood of 1.

4.3.3 Complex powers of numbers near unity

Equation (4.13) raises the question: what if ϵ or x, or both, are complex? Changing the symbol $z \leftarrow x$ and observing that the infinitesimal ϵ may also be complex, one wants to know whether

$$(1+\epsilon)^z \approx 1 + z\epsilon \tag{4.14}$$

still holds. No work we have yet done in the book answers the question, because although a complex infinitesimal ϵ poses no particular problem, the action of a complex power z remains undefined. Still, for consistency's sake, one would like (4.14) to hold. In fact nothing prevents us from defining the action of a complex power such that (4.14) does hold, which we now do, logically extending the known result (4.13) into the new domain.

Section 5.4 will investigate the extremely interesting effects which arise when $\Re(\epsilon) = 0$ and the power z in (4.14) grows large, but for the moment we shall use the equation in a more ordinary manner to develop the concept and basic application of the derivative, as follows.

[4]Other than "the first-order Taylor expansion," but such an unwieldy name does not fit the present context. The Taylor expansion as such will be introduced later (Ch. 8).

4.4 The derivative

With (4.14) in hand, we now stand in a position to introduce the derivative. What is the derivative? The *derivative* is the instantaneous rate or slope of a function. In mathematical symbols and for the moment using real numbers,

$$f'(t) \equiv \lim_{\epsilon\to 0^+} \frac{f(t+\epsilon/2) - f(t-\epsilon/2)}{\epsilon}. \tag{4.15}$$

Alternately, one can define the same derivative in the unbalanced form

$$f'(t) = \lim_{\epsilon\to 0^+} \frac{f(t+\epsilon) - f(t)}{\epsilon},$$

but this book generally prefers the more elegant balanced form (4.15), which we will now use in developing the derivative's several properties through the rest of the chapter.[5]

4.4.1 The derivative of the power series

In the very common case that $f(t)$ is the power series

$$f(t) = \sum_{k=-\infty}^{\infty} c_k t^k, \tag{4.16}$$

where the c_k are in general complex coefficients, (4.15) says that

$$\begin{aligned} f'(t) &= \sum_{k=-\infty}^{\infty} \lim_{\epsilon\to 0^+} \frac{(c_k)(t+\epsilon/2)^k - (c_k)(t-\epsilon/2)^k}{\epsilon} \\ &= \sum_{k=-\infty}^{\infty} \lim_{\epsilon\to 0^+} c_k t^k \frac{(1+\epsilon/2t)^k - (1-\epsilon/2t)^k}{\epsilon}. \end{aligned}$$

Applying (4.14), this is

$$f'(t) = \sum_{k=-\infty}^{\infty} \lim_{\epsilon\to 0^+} c_k t^k \frac{(1+k\epsilon/2t) - (1-k\epsilon/2t)}{\epsilon},$$

which simplifies to

$$f'(t) = \sum_{k=-\infty}^{\infty} c_k k t^{k-1}. \tag{4.17}$$

[5]From this section through § 4.7, the mathematical notation grows a little thick. There is no helping this. The reader is advised to tread through these sections line by stubborn line, in the good trust that the math thus gained will prove both interesting and useful.

Equation (4.17) gives the general derivative of the power series.[6]

4.4.2 The Leibnitz notation

The $f'(t)$ notation used above for the derivative is due to Sir Isaac Newton, and is easier to start with. Usually better on the whole, however, is G.W. Leibnitz's notation[7]

$$\begin{aligned} dt &= \epsilon, \\ df &= f(t + dt/2) - f(t - dt/2), \end{aligned}$$

such that per (4.15),

$$f'(t) = \frac{df}{dt}. \tag{4.18}$$

Here dt is the infinitesimal, and df is a dependent infinitesimal whose size *relative to* dt depends on the independent variable t. For the independent infinitesimal dt, conceptually, one can choose any infinitesimal size ϵ. Usually the exact choice of size doesn't matter, but occasionally when there are two independent variables it helps the analysis to adjust the size of one of the independent infinitesimals with respect to the other.

The meaning of the symbol d unfortunately depends on the context. In (4.18), the meaning is clear enough: $d(\cdot)$ signifies how much $(\cdot)$ changes when the independent variable t increments by dt.[8] Notice, however, that the notation dt itself has two distinct meanings:[9]

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[6] Equation (4.17) admittedly has not explicitly considered what happens when the real t becomes the complex z, but § 4.4.3 will remedy the oversight.

[7] This subsection is likely to confuse many readers the first time they read it. The reason is that Leibnitz elements like dt and ∂f usually tend to appear in practice in certain specific relations to one another, like $\partial f/\partial z$. As a result, many users of applied mathematics have never developed a clear understanding as to precisely what the individual symbols mean. Often they have developed positive misunderstandings. Because there is significant practical benefit in learning how to handle the Leibnitz notation correctly—particularly in applied complex variable theory—this subsection seeks to present each Leibnitz element in its correct light.

[8] If you do not fully understand this sentence, reread it carefully with reference to (4.15) and (4.18) until you do; it's important.

[9] This is difficult, yet the author can think of no clearer, more concise way to state it. The quantities dt and df represent coordinated infinitesimal changes in t and f respectively, so there is usually no trouble with treating dt and df as though they were the same kind of thing. However, at the fundamental level they really aren't.

If t is an independent variable, then dt is just an infinitesimal of some kind, whose specific size could be a function of t but more likely is just a constant. If a constant, then dt does not fundamentally have anything to do with t as such. In fact, if s and t

- the independent infinitesimal $dt = \epsilon$; and
- $d(t)$, which is how much (t) changes as t increments by dt.

At first glance, the distinction between dt and $d(t)$ seems a distinction without a difference; and for most practical cases of interest, so indeed it is. However, when switching perspective in mid-analysis as to which variables are dependent and which are independent, or when changing multiple independent complex variables simultaneously, the math can get a little tricky. In such cases, it may be wise to use the symbol dt to mean $d(t)$ only, introducing some unambiguous symbol like ϵ to represent the independent infinitesimal. In any case you should appreciate the conceptual difference between $dt = \epsilon$ and $d(t)$.

---

are both independent variables, then we can (and in complex analysis sometimes do) say that $ds = dt = \epsilon$, after which nothing prevents us from using the symbols ds and dt interchangeably. Maybe it would be clearer in some cases to write ϵ instead of dt, but the latter is how it is conventionally written.

By contrast, if f is a dependent variable, then df or $d(f)$ is the amount by which f changes as t changes by dt. The df is infinitesimal but not constant; it is a function of t. Maybe it would be clearer in some cases to write $d_t f$ instead of df, but for most cases the former notation is unnecessarily cluttered; the latter is how it is conventionally written.

Now, most of the time, what we are interested in is not dt or df as such, but rather the ratio df/dt or the sum $\sum_k f(k\,dt)\,dt = \int f(t)\,dt$. For this reason, we do not usually worry about which of df and dt is the independent infinitesimal, nor do we usually worry about the precise value of dt. This leads one to forget that dt does indeed have a precise value. What confuses is when one changes perspective in mid-analysis, now regarding f as the independent variable. Changing perspective is allowed and perfectly proper, but one must take care: the dt and df after the change are not the same as the dt and df before the change. However, the ratio df/dt remains the same in any case.

Sometimes when writing a differential equation like the potential-kinetic energy equation $ma\,dx = mv\,dv$, we do not necessarily have either v or x in mind as the independent variable. This is fine. The important point is that dv and dx be coordinated so that the ratio dv/dx has a definite value no matter which of the two be regarded as independent, or whether the independent be some third variable (like t) not in the equation.

One can avoid the confusion simply by keeping the dv/dx or df/dt always in ratio, never treating the infinitesimals individually. Many applied mathematicians do precisely that. That is okay as far as it goes, but it really denies the entire point of the Leibnitz notation. One might as well just stay with the Newton notation in that case. Instead, this writer recommends that you learn the Leibnitz notation properly, developing the ability to treat the infinitesimals individually.

Because the book is a book of applied mathematics, this footnote does not attempt to say everything there is to say about infinitesimals. For instance, it has not yet pointed out (but does so now) that even if s and t are equally independent variables, one can have $dt = \epsilon(t)$, $ds = \delta(s,t)$, such that dt has prior independence to ds. The point is not to fathom all the possible implications from the start; you can do that as the need arises. The point is to develop a clear picture in your mind of what a Leibnitz infinitesimal really is. Once you have the picture, you can go from there.

Where two or more independent variables are at work in the same equation, it is conventional to use the symbol ∂ instead of d, as a reminder that the reader needs to pay attention to which ∂ tracks which independent variable.[10] (If needed or desired, one can write $\partial_t(\cdot)$ when tracking t, $\partial_s(\cdot)$ when tracking s, etc. You have to be careful, though. Such notation appears only rarely in the literature, so your audience might not understand it when you write it.) Conventional shorthand for $d(df)$ is d^2f; for $(dt)^2$, dt^2; so

$$\frac{d(df/dt)}{dt} = \frac{d^2f}{dt^2}$$

is a derivative of a derivative, or *second derivative.* By extension, the notation

$$\frac{d^kf}{dt^k}$$

represents the kth derivative.

4.4.3 The derivative of a function of a complex variable

For (4.15) to be robust, written here in the slightly more general form

$$\frac{df}{dz} = \lim_{\epsilon\to 0} \frac{f(z+\epsilon/2) - f(z-\epsilon/2)}{\epsilon}, \tag{4.19}$$

one should like it to evaluate the same in the limit regardless of the complex phase of ϵ. That is, if δ is a positive real infinitesimal, then it should be equally valid to let $\epsilon = \delta$, $\epsilon = -\delta$, $\epsilon = i\delta$, $\epsilon = -i\delta$, $\epsilon = (4 - i3)\delta$ or any other infinitesimal value, so long as $0 < |\epsilon| \ll 1$. One should like the derivative (4.19) to come out the same regardless of the Argand direction from which ϵ approaches 0 (see Fig. 2.5). In fact for the sake of robustness, one normally demands that derivatives do come out the same regardless of the Argand direction; and (4.19) rather than (4.15) is the definition we normally use for the derivative for this reason. Where the limit (4.19) is sensitive to the Argand direction or complex phase of ϵ, there we normally say that the derivative does not exist.

Where the derivative (4.19) does exist—where the derivative is finite and insensitive to Argand direction—there we say that the function $f(z)$ is *differentiable.*

Excepting the nonanalytic parts of complex numbers ($|\cdot|$, $\arg(\cdot)$, $(\cdot)^*$, $\Re(\cdot)$ and $\Im(\cdot)$; see § 2.12.3), plus the Heaviside unit step $u(t)$ and the Dirac

[10] The writer confesses that he remains unsure why this minor distinction merits the separate symbol ∂, but he accepts the notation as conventional nevertheless.

delta $\delta(t)$ (§ 7.7), most functions encountered in applications do meet the criterion (4.19) except at isolated nonanalytic points (like $z = 0$ in $h(z) = 1/z$ or $g(z) = \sqrt{z}$). Meeting the criterion, such functions are fully differentiable except at their poles (where the derivative goes infinite in any case) and other nonanalytic points. Particularly, the key formula (4.14), written here as

$$(1+\epsilon)^w \approx 1 + w\epsilon,$$

works without modification when ϵ is complex; so the derivative (4.17) of the general power series,

$$\frac{d}{dz}\sum_{k=-\infty}^{\infty} c_k z^k = \sum_{k=-\infty}^{\infty} c_k k z^{k-1} \tag{4.20}$$

holds equally well for complex z as for real.

4.4.4 The derivative of z^a

Inspection of § 4.4.1's logic in light of (4.14) reveals that nothing prevents us from replacing the real t, real ϵ and integral k of that section with arbitrary complex z, ϵ and a. That is,

$$\begin{aligned}
\frac{d(z^a)}{dz} &= \lim_{\epsilon\to 0}\frac{(z+\epsilon/2)^a - (z-\epsilon/2)^a}{\epsilon} \\
&= \lim_{\epsilon\to 0} z^a \frac{(1+\epsilon/2z)^a - (1-\epsilon/2z)^a}{\epsilon} \\
&= \lim_{\epsilon\to 0} z^a \frac{(1+a\epsilon/2z) - (1-a\epsilon/2z)}{\epsilon},
\end{aligned}$$

which simplifies to

$$\frac{d(z^a)}{dz} = az^{a-1} \tag{4.21}$$

for any complex z and a.

How exactly to evaluate z^a or z^{a-1} when a is complex is another matter, treated in § 5.4 and its (5.12); but in any case you can use (4.21) for real a right now.

4.4.5 The logarithmic derivative

Sometimes one is more interested in knowing the rate of $f(t)$ *relative to the value of* $f(t)$ than in knowing the absolute rate itself. For example, if you inform me that you earn \$ 1000 a year on a bond you hold, then I may commend you vaguely for your thrift but otherwise the information does not tell me much. However, if you inform me instead that you earn 10 percent a year on the same bond, then I might want to invest. The latter figure is a relative rate, or *logarithmic derivative,*

$$\frac{df/dt}{f(t)} = \frac{d}{dt}\ln f(t). \tag{4.22}$$

The investment principal grows at the absolute rate df/dt, but the bond's interest rate is $(df/dt)/f(t)$.

The natural logarithmic notation $\ln f(t)$ may not mean much to you yet, as we'll not introduce it formally until § 5.2, so you can ignore the right side of (4.22) for the moment; but the equation's left side at least should make sense to you. It expresses the significant concept of a relative rate, like 10 percent annual interest on a bond.

4.5 Basic manipulation of the derivative

This section introduces the derivative chain and product rules.

4.5.1 The derivative chain rule

If f is a function of w, which itself is a function of z, then[11]

$$\frac{df}{dz} = \left(\frac{df}{dw}\right)\left(\frac{dw}{dz}\right). \tag{4.23}$$

[11]For example, one can rewrite

$$f(z) = \sqrt{3z^2 - 1}$$

in the form

$$\begin{aligned} f(w) &= w^{1/2}, \\ w(z) &= 3z^2 - 1. \end{aligned}$$

Then

$$\begin{aligned} \frac{df}{dw} &= \frac{1}{2w^{1/2}} = \frac{1}{2\sqrt{3z^2-1}}, \\ \frac{dw}{dz} &= 6z, \end{aligned}$$

so by (4.23),

$$\frac{df}{dz} = \left(\frac{df}{dw}\right)\left(\frac{dw}{dz}\right) = \frac{6z}{2\sqrt{3z^2-1}} = \frac{3z}{\sqrt{3z^2-1}}.$$

Equation (4.23) is the *derivative chain rule.*[12]

4.5.2 The derivative product rule

In general per (4.19),

$$d\left[\prod_k f_k(z)\right] = \prod_k f_k\left(z + \frac{dz}{2}\right) - \prod_k f_k\left(z - \frac{dz}{2}\right).$$

But to first order,

$$f_k\left(z \pm \frac{dz}{2}\right) \approx f_k(z) \pm \left(\frac{df_k}{dz}\right)\left(\frac{dz}{2}\right) = f_k(z) \pm \frac{df_k}{2},$$

so in the limit,

$$d\left[\prod_k f_k(z)\right] = \prod_k \left(f_k(z) + \frac{df_k}{2}\right) - \prod_k \left(f_k(z) - \frac{df_k}{2}\right).$$

Since the product of two or more df_k is negligible compared to the first-order infinitesimals to which they are added here, this simplifies to

$$d\left[\prod_k f_k(z)\right] = \left[\prod_k f_k(z)\right]\left[\sum_j \frac{df_j}{2f_j(z)}\right] - \left[\prod_k f_k(z)\right]\left[\sum_j \frac{-df_j}{2f_j(z)}\right],$$

or in other words

$$d\prod_k f_k = \left[\prod_k f_k\right]\left[\sum_j \frac{df_j}{f_j}\right]. \tag{4.24}$$

In the common case of only two f_k, this comes to

$$d(f_1 f_2) = f_2\,df_1 + f_1\,df_2. \tag{4.25}$$

Equation (4.24) is the *derivative product rule.*

After studying the complex exponential in Ch. 5, we shall stand in a position to write (4.24) in the slightly specialized but often useful form[13]

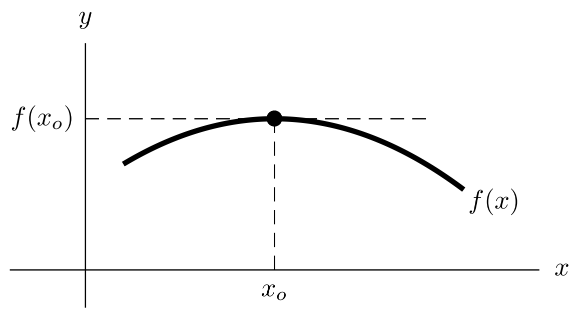
$$d\left[\prod_k g_k^{a_k} \prod_k e^{b_k h_k}\right] = \left[\prod_k g_k^{a_k} \prod_k e^{b_k h_k}\right]\left[\sum_j a_j \frac{dg_j}{g_j} + \sum_j b_j\,dh_j\right]. \tag{4.26}$$

---

[12] It bears emphasizing to readers who may inadvertently have picked up unhelpful ideas about the Leibnitz notation in the past: the dw factor in the denominator cancels the dw factor in the numerator; a thing divided by itself is 1. That's it. There is nothing more to the proof of the derivative chain rule than that.

[13] This paragraph is extra. You can skip it for now if you prefer.

Figure 4.3: A local extremum.



where the a_k and b_k are arbitrary complex coefficients and the g_k and h_k are arbitrary functions.[14]

4.6 Extrema and higher derivatives

One problem which arises very frequently in applied mathematics is the problem of finding a local *extremum*—that is, a local minimum or maximum—of a real-valued function $f(x)$. Refer to Fig. 4.3. The almost distinctive characteristic of the extremum x_o is that[15]

$$\left.\frac{df}{dx}\right|_{x=x_o} = 0. \tag{4.27}$$

At the extremum, the slope is zero. The curve momentarily runs level there. One solves (4.27) to find the extremum.

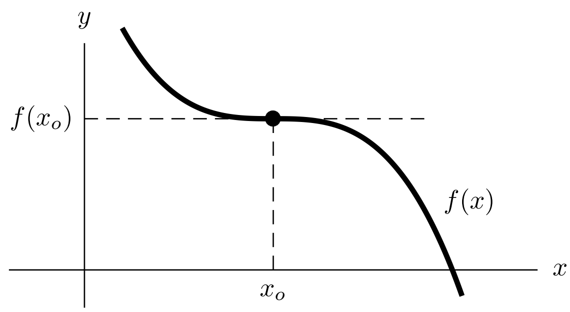
Whether the extremum be a minimum or a maximum depends on whether the curve turn from a downward slope to an upward, or from an upward

[14]The subsection is sufficiently abstract that it is a little hard to understand unless one already knows what it means. An example may help:

$$d\left[\frac{u^2v^3}{z}e^{-5t}\right] = \left[\frac{u^2v^3}{z}e^{-5t}\right]\left[2\frac{du}{u} + 3\frac{dv}{v} - \frac{dz}{z} - 5\,dt\right].$$

[15]The notation $P|_Q$ means "P when Q," "P, given Q," or "P evaluated at Q." Sometimes it is alternately written $P|Q$.

Figure 4.4: A level inflection.



slope to a downward, respectively. If from downward to upward, then the derivative of the slope is evidently positive; if from upward to downward, then negative. But the derivative of the slope is just the derivative of the derivative, or second derivative. Hence if $df/dx = 0$ at $x = x_o$, then

$$\begin{aligned} \left.\frac{d^2 f}{dx^2}\right|_{x=x_o} &> 0 \text{ implies a local minimum at } x_o; \\ \left.\frac{d^2 f}{dx^2}\right|_{x=x_o} &< 0 \text{ implies a local maximum at } x_o. \end{aligned}$$

Regarding the case

$$\left.\frac{d^2 f}{dx^2}\right|_{x=x_o} = 0,$$

this might be either a minimum or a maximum but more probably is neither, being rather a *level inflection point* as depicted in Fig. 4.4.[16] (In general the term *inflection point* signifies a point at which the second derivative is zero. The inflection point of Fig. 4.4 is *level* because its first derivative is zero, too.)

[16] Of course if the first and second derivatives are zero not just at $x = x_o$ but everywhere, then $f(x) = y_o$ is just a level straight line, but you knew that already. Whether one chooses to call some random point on a level straight line an inflection point or an extremum, or both or neither, would be a matter of definition, best established not by prescription but rather by the needs of the model at hand.

4.7 L'Hôpital's rule

If $z = z_o$ is a root of both $f(z)$ and $g(z)$, or alternately if $z = z_o$ is a singularity of both functions,[17] then *l'Hôpital's rule* holds that

$$\lim_{z\to z_o} \frac{f(z)}{g(z)} = \left.\frac{df/dz}{dg/dz}\right|_{z=z_o}. \tag{4.28}$$

In the case where $z = z_o$ is a root, l'Hôpital's rule is proven by reasoning[18]

$$\begin{aligned}\lim_{z\to z_o} \frac{f(z)}{g(z)} &= \lim_{z\to z_o} \frac{f(z)-0}{g(z)-0} \\ &= \lim_{z\to z_o} \frac{f(z)-f(z_o)}{g(z)-g(z_o)} = \lim_{z\to z_o} \frac{df}{dg} = \lim_{z\to z_o} \frac{df/dz}{dg/dz}.\end{aligned}$$

In the case where $z = z_o$ is a singularity, new functions $F(z) \equiv 1/f(z)$ and $G(z) \equiv 1/g(z)$ of which $z = z_o$ is a root are defined, with which

$$\lim_{z\to z_o} \frac{f(z)}{g(z)} = \lim_{z\to z_o} \frac{G(z)}{F(z)} = \lim_{z\to z_o} \frac{dG}{dF} = \lim_{z\to z_o} \frac{-dg/g^2}{-df/f^2},$$

where we have used the fact from (4.21) that $d(1/u) = -du/u^2$ for any u. Canceling the minus signs and multiplying by g^2/f^2, we have that

$$\lim_{z\to z_o} \frac{g(z)}{f(z)} = \lim_{z\to z_o} \frac{dg}{df}.$$

Inverting,

$$\lim_{z\to z_o} \frac{f(z)}{g(z)} = \lim_{z\to z_o} \frac{df}{dg} = \lim_{z\to z_o} \frac{df/dz}{dg/dz}.$$

What about the case where z_o is infinite? In this case, we define the new variable $Z = 1/z$ and the new functions $\Phi(Z) = f(1/Z) = f(z)$ and $\Gamma(Z) = g(1/Z) = g(z)$, with which we can apply L'Hôpital's rule for $Z \to 0$ to obtain

$$\begin{aligned}\lim_{z\to z_o} \frac{f(z)}{g(z)} &= \lim_{Z\to 0} \frac{\Phi(Z)}{\Gamma(Z)} = \lim_{Z\to 0} \frac{d\Phi/dZ}{d\Gamma/dZ} = \lim_{Z\to 0} \frac{d\left[f\left(\frac{1}{Z}\right)\right]/dZ}{d\left[g\left(\frac{1}{Z}\right)\right]/dZ} \\ &= \lim_{Z\to 0} \frac{(df/dz)(dz/dZ)}{(dg/dz)(dz/dZ)} = \lim_{z\to z_o} \frac{(df/dz)(-z^2)}{(dg/dz)(-z^2)} = \lim_{z\to z_o} \frac{df/dz}{dg/dz}.\end{aligned}$$

Nothing in the derivation requires that z or z_o be real.

[17] See § 2.11 for the definitions of *root* and *singularity*.

[18] Partly with reference to [3, "L'Hopital's rule," 03:40, 5 April 2006].

L'Hôpital's rule is used in evaluating indeterminate forms of the kinds $0/0$ and ∞/∞, plus related forms like $(0)(\infty)$ which can be recast into either of the two main forms. Good examples of the use require math from Ch. 5 and later, but if we may borrow from (5.7) the natural logarithmic function and its derivative,[19]

$$\frac{d}{dx}\ln x = \frac{1}{x},$$

then a typical L'Hôpital example is [22, § 10-2]

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$$

This shows that natural logarithms grow slower than square roots.

Section 5.3 will put l'Hôpital's rule to work.

4.8 The Newton-Raphson iteration

The *Newton-Raphson iteration* is a powerful, fast converging, broadly applicable method for finding roots numerically. Given a function $f(z)$ of which the root is desired, the Newton-Raphson iteration is

$$z_{k+1} = z_k - \left.\frac{f(z)}{\frac{d}{dz}f(z)}\right|_{z=z_k}. \tag{4.29}$$

One begins the iteration by guessing the root and calling the guess z_0. Then z_1, z_2, z_3, etc., calculated in turn by the iteration (4.29), give successively better estimates of the true root z_∞.

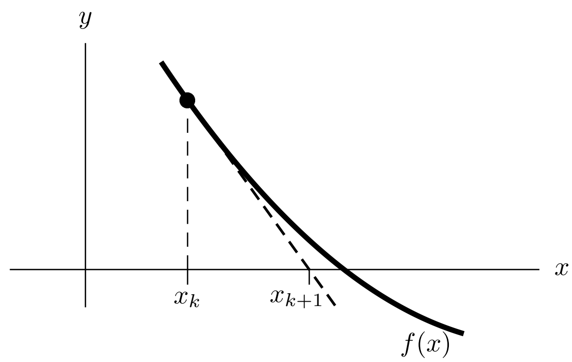
To understand the Newton-Raphson iteration, consider the function $y = f(x)$ of Fig 4.5. The iteration approximates the curve $f(x)$ by its tangent line[20] (shown as the dashed line in the figure):

$$\tilde{f}_k(x) = f(x_k) + \left[\frac{d}{dx}f(x)\right]_{x=x_k}(x - x_k).$$

[19] This paragraph is optional reading for the moment. You can read Ch. 5 first, then come back here and read the paragraph if you prefer.

[20] A *tangent* line, also just called a *tangent,* is the line which most nearly approximates a curve at a given point. The tangent touches the curve at the point, and in the neighborhood of the point it goes in the same direction the curve goes. The dashed line in Fig. 4.5 is a good example of a tangent line.

Figure 4.5: The Newton-Raphson iteration.



It then approximates the root x_{k+1} as the point at which $\tilde{f}_k(x_{k+1}) = 0$:

$$\tilde{f}_k(x_{k+1}) = 0 = f(x_k) + \left[\frac{d}{dx}f(x)\right]_{x=x_k}(x_{k+1} - x_k).$$

Solving for x_{k+1}, we have that

$$x_{k+1} = x_k - \left.\frac{f(x)}{\frac{d}{dx}f(x)}\right|_{x=x_k},$$

which is (4.29) with $x \leftarrow z$.

Although the illustration uses real numbers, nothing forbids complex z and $f(z)$. The Newton-Raphson iteration works just as well for these.

The principal limitation of the Newton-Raphson arises when the function has more than one root, as most interesting functions do. The iteration often but not always converges on the root nearest the initial guess z_o; but in any case there is no guarantee that the root it finds is the one you wanted. The most straightforward way to beat this problem is to find *all* the roots: first you find some root α, then you remove that root (without affecting any of the other roots) by dividing $f(z)/(z - \alpha)$, then you find the next root by iterating on the new function $f(z)/(z - \alpha)$, and so on until you have found all the roots. If this procedure is not practical (perhaps because the function has a large or infinite number of roots), then you should probably take care to make a sufficiently accurate initial guess if you can.

A second limitation of the Newton-Raphson is that if one happens to guess z_0 badly, the iteration may never converge on the root. For example, the roots of $f(z) = z^2 + 2$ are $z = \pm i\sqrt{2}$, but if you guess that $z_0 = 1$ then the iteration has no way to leave the real number line, so it never converges. You fix this by setting z_0 to a complex initial guess.

A third limitation arises where there is a multiple root. In this case, the Newton-Raphson normally still converges, but relatively slowly. For instance, the Newton-Raphson converges relatively slowly on the triple root of $f(z) = z^3$.

Usually in practice, the Newton-Raphson works very well. For most functions, once the Newton-Raphson finds the root's neighborhood, it converges on the actual root remarkably fast. Figure 4.5 shows why: in the neighborhood, the curve hardly departs from the straight line.

The Newton-Raphson iteration is a champion square root calculator, incidentally. Consider

$$f(x) = x^2 - p,$$

whose roots are

$$x = \pm\sqrt{p}.$$

Per (4.29), the Newton-Raphson iteration for this is

$$x_{k+1} = \frac{1}{2}\left[x_k + \frac{p}{x_k}\right]. \tag{4.30}$$

If you start by guessing

$$x_0 = 1$$

and iterate several times, the iteration (4.30) converges on $x_\infty = \sqrt{p}$ fast. To calculate the nth root $x = p^{1/n}$, let

$$f(x) = x^n - p$$

and iterate[21]

$$x_{k+1} = \frac{1}{n}\left[(n-1)x_k + \frac{p}{x_k^{n-1}}\right]. \tag{4.31}$$

(References: [22, § 4-9]; [18, § 6.1.1]; [28].)

Chapter 8 continues the general discussion of the derivative, treating the Taylor series.

[21]Equations (4.30) and (4.31) converge successfully toward $x = p^{1/n}$ for most complex p and x_0. Given $x_0 = 1$, however, they converge reliably and orderly only for real, nonnegative p. (To see why, sketch $f(x)$ in the fashion of Fig. 4.5.)

If reliable, orderly convergence is needed for complex $p = u + iv = \sigma \operatorname{cis} \psi$, $\sigma \geq 0$, you can decompose $p^{1/n}$ per de Moivre's theorem (3.26) as $p^{1/n} = \sigma^{1/n} \operatorname{cis}(\psi/n)$, in which $\operatorname{cis}(\psi/n) = \cos(\psi/n) + i\sin(\psi/n)$ is calculated by the Taylor series of Table 8.1. Then σ is real and nonnegative, so (4.30) or (4.31) converges toward $\sigma^{1/n}$ in a reliable and orderly manner.

The Newton-Raphson iteration however excels as a *practical* root-finding technique, so it often pays to be a little less theoretically rigid in applying it. When the iteration does not seem to converge, the best tactic may simply be to start over again with some randomly chosen complex x_0. This usually works, and it saves a lot of trouble.

Chapter 5

The complex exponential

In higher mathematics, the complex exponential is almost impossible to avoid. It seems to appear just about everywhere. This chapter introduces the concept of the natural exponential and of its inverse, the natural logarithm; and shows how the two operate on complex arguments. It derives the functions' basic properties and explains their close relationship to the trigonometrics. It works out the functions' derivatives and the derivatives of the basic members of the trigonometric and inverse trigonometric families to which they respectively belong.

5.1 The real exponential

Consider the factor

$$(1+\epsilon)^N.$$

This is the overall factor by which a quantity grows after N iterative rounds of multiplication by $(1+\epsilon)$. What happens when ϵ is very small but N is very large? The really interesting question is, what happens in the limit, as $\epsilon \to 0$ and $N \to \infty$, while $x = \epsilon N$ remains a finite number? The answer is that the factor becomes

$$\exp x \equiv \lim_{\epsilon \to 0^+} (1+\epsilon)^{x/\epsilon}. \tag{5.1}$$

Equation (5.1) is the definition of the *exponential function* or *natural exponential function.* We can alternately write the same definition in the form

$$\begin{aligned} \exp x &= e^x, & (5.2)\\ e &\equiv \lim_{\epsilon \to 0^+} (1+\epsilon)^{1/\epsilon}. & (5.3) \end{aligned}$$

Whichever form we write it in, the question remains as to whether the limit actually exists; that is, whether $0 < e < \infty$; whether in fact we can put some concrete bound on e. To show that we can,[1] we observe per (4.15)[2] that the derivative of the exponential function is

$$\begin{aligned}
\frac{d}{dx}\exp x &= \lim_{\delta\to 0^+} \frac{\exp(x+\delta/2) - \exp(x-\delta/2)}{\delta} \\
&= \lim_{\delta,\epsilon\to 0^+} \frac{(1+\epsilon)^{(x+\delta/2)/\epsilon} - (1+\epsilon)^{(x-\delta/2)/\epsilon}}{\delta} \\
&= \lim_{\delta,\epsilon\to 0^+} (1+\epsilon)^{x/\epsilon}\frac{(1+\epsilon)^{+\delta/2\epsilon} - (1+\epsilon)^{-\delta/2\epsilon}}{\delta} \\
&= \lim_{\delta,\epsilon\to 0^+} (1+\epsilon)^{x/\epsilon}\frac{(1+\delta/2) - (1-\delta/2)}{\delta} \\
&= \lim_{\epsilon\to 0^+} (1+\epsilon)^{x/\epsilon},
\end{aligned}$$

which is to say that

$$\frac{d}{dx}\exp x = \exp x. \tag{5.4}$$

This is a curious, important result: the derivative of the exponential function is the exponential function itself; the slope and height of the exponential function are everywhere equal. For the moment, however, what interests us is that

$$\frac{d}{dx}\exp 0 = \exp 0 = \lim_{\epsilon\to 0^+} (1+\epsilon)^0 = 1,$$

which says that the slope and height of the exponential function are both unity at $x = 0$, implying that the straight line which best approximates the exponential function in that neighborhood—the *tangent line,* which just grazes the curve—is

$$y(x) = 1 + x.$$

With the tangent line $y(x)$ found, the next step toward putting a concrete bound on e is to show that $y(x) \le \exp x$ for all real x, that the curve runs nowhere below the line. To show this, we observe per (5.1) that the essential action of the exponential function is to multiply repeatedly by $1+\epsilon$ as x

[1]Excepting (5.4), the author would prefer to omit much of the rest of this section, but even at the applied level cannot think of a logically permissible way to do it. It seems nonobvious that the limit $\lim_{\epsilon\to 0^+}(1+\epsilon)^{1/\epsilon}$ actually does exist. The rest of this section shows why it does.

[2]Or per (4.19). It works the same either way.

increases, to divide repeatedly by $1 + \epsilon$ as x decreases. Since $1 + \epsilon > 1$, this action means for real x that

$$\exp x_1 \le \exp x_2 \quad \text{if } x_1 \le x_2.$$

However, a positive number remains positive no matter how many times one multiplies or divides it by $1 + \epsilon$, so the same action also means that

$$0 \le \exp x$$

for all real x. In light of (5.4), the last two equations imply further that

$$\begin{aligned} \frac{d}{dx}\exp x_1 &\le \frac{d}{dx}\exp x_2 \quad \text{if } x_1 \le x_2, \\ 0 &\le \frac{d}{dx}\exp x. \end{aligned}$$

But we have purposely defined the line $y(x) = 1 + x$ such that

$$\begin{aligned} \exp 0 &= \quad y(0) &= 1, \\ \frac{d}{dx}\exp 0 &= \frac{d}{dx}y(0) &= 1; \end{aligned}$$

that is, such that the curve of $\exp x$ just grazes the line $y(x)$ at $x = 0$. Rightward, at $x > 0$, evidently the curve's slope only increases, bending upward away from the line. Leftward, at $x < 0$, evidently the curve's slope only decreases, again bending upward away from the line. Either way, the curve never crosses below the line for real x. In symbols,

$$y(x) \le \exp x.$$

Figure 5.1 depicts.

Evaluating the last inequality at $x = -1/2$ and $x = 1$, we have that

$$\begin{aligned} \frac{1}{2} &\le \exp\left(-\frac{1}{2}\right), \\ 2 &\le \exp(1). \end{aligned}$$

But per (5.2), $\exp x = e^x$, so

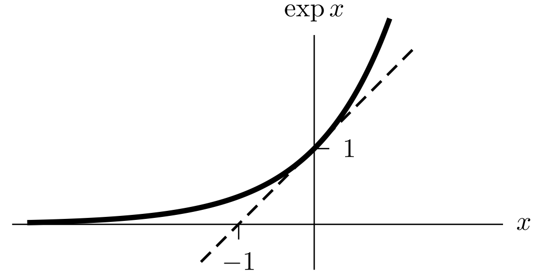
$$\begin{aligned} \frac{1}{2} &\le e^{-1/2}, \\ 2 &\le e^1, \end{aligned}$$

or in other words,

$$2 \le e \le 4, \tag{5.5}$$

which in consideration of (5.2) puts the desired bound on the exponential function. The limit does exist.

Figure 5.1: The natural exponential.



5.2 The natural logarithm

In the general exponential expression a^x, one can choose any base a; for example, $a = 2$ is an interesting choice. As we shall see in § 5.4, however, it turns out that $a = e$, where e is the constant introduced in (5.3), is the most interesting choice of all. For this reason among others, the base-e logarithm is similarly interesting, such that we define for it the special notation

$$\ln(\cdot) = \log_e(\cdot),$$

and call it the *natural logarithm.* Just as for any other base a, so also for base $a = e$; thus the natural logarithm inverts the natural exponential and vice versa:

$$\begin{aligned} \ln \exp x &= \ln e^x &= x, \\ \exp \ln x &= e^{\ln x} &= x. \end{aligned} \tag{5.6}$$

Figure 5.2 plots the natural logarithm.

If

$$y = \ln x,$$

then

$$x = \exp y,$$

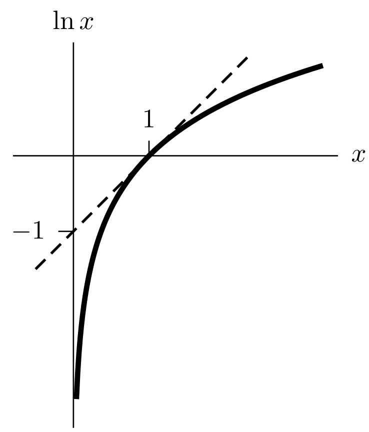
and per (5.4),

$$\frac{dx}{dy} = \exp y.$$

But this means that

$$\frac{dx}{dy} = x,$$

Figure 5.2: The natural logarithm.



the inverse of which is

$$\frac{dy}{dx} = \frac{1}{x}.$$

In other words,

$$\frac{d}{dx} \ln x = \frac{1}{x}. \tag{5.7}$$

Like many of the equations in these early chapters, here is another rather significant result.[3]

5.3 Fast and slow functions

The exponential $\exp x$ is a *fast function.* The logarithm $\ln x$ is a *slow function.* These functions grow, diverge or decay respectively faster and slower than x^a.

Such claims are proven by l'Hôpital's rule (4.28). Applying the rule, we

[3]Besides the result itself, the technique which leads to the result is also interesting and is worth mastering. We shall use the technique more than once in this book.

have that

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x}{x^a} = \lim_{x\to\infty}\frac{-1}{ax^a} &= \begin{cases}0 & \text{if } a>0,\\ +\infty & \text{if } a\le 0,\end{cases}\\ \lim_{x\to 0^+}\frac{\ln x}{x^a} = \lim_{x\to 0^+}\frac{-1}{ax^a} &= \begin{cases}-\infty & \text{if } a\ge 0,\\ 0 & \text{if } a<0,\end{cases}\end{aligned} \tag{5.8}$$

which reveals the logarithm to be a slow function. Since the $\exp(\cdot)$ and $\ln(\cdot)$ functions are mutual inverses, we can leverage (5.8) to show also that

$$\begin{aligned}\lim_{x\to\infty}\frac{\exp(\pm x)}{x^a} &= \lim_{x\to\infty}\exp\left[\ln\frac{\exp(\pm x)}{x^a}\right]\\ &= \lim_{x\to\infty}\exp\left[\pm x - a\ln x\right]\\ &= \lim_{x\to\infty}\exp\left[(x)\left(\pm 1 - a\frac{\ln x}{x}\right)\right]\\ &= \lim_{x\to\infty}\exp\left[(x)(\pm 1 - 0)\right]\\ &= \lim_{x\to\infty}\exp\left[\pm x\right].\end{aligned}$$

That is,

$$\begin{aligned}\lim_{x\to\infty}\frac{\exp(+x)}{x^a} &= \infty,\\ \lim_{x\to\infty}\frac{\exp(-x)}{x^a} &= 0,\end{aligned} \tag{5.9}$$

which reveals the exponential to be a fast function. Exponentials grow or decay faster than powers; logarithms diverge slower.

Such conclusions are extended to bases other than the natural base e simply by observing that $\log_b x = \ln x/\ln b$ and that $b^x = \exp(x\ln b)$. Thus exponentials generally are fast and logarithms generally are slow, regardless of the base.[4]

5.4 Euler's formula

The result of § 5.1 leads to one of the central questions in all of mathematics. How can one evaluate

$$\exp i\theta = \lim_{\epsilon\to 0^+}(1+\epsilon)^{i\theta/\epsilon},$$

[4]There are of course some degenerate edge cases like $a = 0$ and $b = 1$. The reader can detail these as the need arises.

where $i^2 = -1$ is the imaginary unit introduced in § 2.12?

To begin, one can take advantage of (4.14) to write the last equation in the form

$$\exp i\theta = \lim_{\epsilon \to 0^+} (1 + i\epsilon)^{\theta/\epsilon},$$

but from here it is not obvious where to go. The book's development up to the present point gives no obvious direction. In fact it appears that the interpretation of $\exp i\theta$ remains for us to define, if we can find a way to define it which fits sensibly with our existing notions of the real exponential. So, if we don't quite know where to go with this yet, what do we know?

One thing we know is that if $\theta = \epsilon$, then

$$\exp(i\epsilon) = (1 + i\epsilon)^{\epsilon/\epsilon} = 1 + i\epsilon.$$

But per § 5.1, the essential operation of the exponential function is to multiply repeatedly by some factor, where the factor is not quite exactly unity—in this case, by $1 + i\epsilon$. So let us multiply a complex number $z = x + iy$ by $1 + i\epsilon$, obtaining

$$(1 + i\epsilon)(x + iy) = (x - \epsilon y) + i(y + \epsilon x).$$

The resulting change in z is

$$\Delta z = (1 + i\epsilon)(x + iy) - (x + iy) = (\epsilon)(-y + ix),$$

in which it is seen that

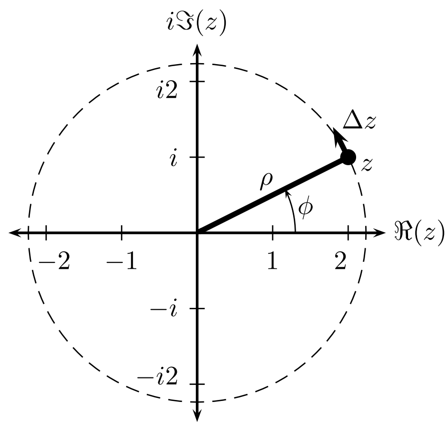
$$\begin{aligned} |\Delta z| &= (\epsilon)\sqrt{y^2 + x^2} &&= (\epsilon)|z|; \\ \arg(\Delta z) &= \arctan \frac{x}{-y} &&= \arg z + \frac{2\pi}{4}. \end{aligned}$$

In other words, *the change is proportional to the magnitude of z, but at a right angle to z's arm in the complex plane.* Refer to Fig. 5.3.

Since the change is directed neither inward nor outward from the dashed circle in the figure, evidently its effect is to move z a distance $(\epsilon)|z|$ counterclockwise about the circle. In other words, referring to the figure, the change Δz leads to

$$\begin{aligned} \Delta\rho &= 0, \\ \Delta\phi &= \epsilon. \end{aligned}$$

Figure 5.3: The complex exponential and Euler's formula.



These two equations—implying an unvarying, steady counterclockwise progression about the circle in the diagram—are somewhat unexpected. Given the steadiness of the progression, it follows from the equations that

$$\begin{aligned} |\exp i\theta| &= \lim_{\epsilon\to 0^+} \left|(1+i\epsilon)^{\theta/\epsilon}\right| &&= |\exp 0| + \lim_{\epsilon\to 0^+} \frac{\theta}{\epsilon}\Delta\rho &&= 1; \\ \arg\left[\exp i\theta\right] &= \lim_{\epsilon\to 0^+} \arg\left[(1+i\epsilon)^{\theta/\epsilon}\right] &&= \arg\left[\exp 0\right] + \lim_{\epsilon\to 0^+} \frac{\theta}{\epsilon}\Delta\phi &&= \theta. \end{aligned}$$

That is, $\exp i\theta$ is the complex number which lies on the Argand unit circle at phase angle θ. In terms of sines and cosines of θ, such a number can be written

$$\exp i\theta = \cos\theta + i\sin\theta = \operatorname{cis}\theta, \tag{5.10}$$

where $\operatorname{cis}(\cdot)$ is as defined in § 3.11. Or, since $\phi \equiv \arg(\exp i\theta) = \theta$,

$$\exp i\phi = \cos\phi + i\sin\phi = \operatorname{cis}\phi. \tag{5.11}$$

Along with the Pythagorean theorem (2.42) and the fundamental theorem of calculus (7.2), (5.11) is perhaps one of the three most famous results in all of mathematics. It is called *Euler's formula,*[5,6] and it opens the exponential domain fully to complex numbers, not just for the natural base e

[5] For native English speakers who do not speak German, Leonhard Euler's name is pronounced as "oiler."

[6] An alternate derivation of Euler's formula (5.11)—less intuitive and requiring slightly

but for any base. To see this, consider in light of Fig. 5.3 and (5.11) that one can express any complex number in the form

$$z = x + iy = \rho \exp i\phi.$$

If a complex base w is similarly expressed in the form

$$w = u + iv = \sigma \exp i\psi,$$

then it follows that

$$\begin{aligned} w^z &= \exp[\ln w^z] \\ &= \exp[z \ln w] \\ &= \exp[(x + iy)(i\psi + \ln \sigma)] \\ &= \exp[(x \ln \sigma - \psi y) + i(y \ln \sigma + \psi x)]. \end{aligned}$$

Since $\exp(\alpha + \beta) = e^{\alpha+\beta} = \exp \alpha \exp \beta$, the last equation is

$$w^z = \exp(x \ln \sigma - \psi y) \exp i(y \ln \sigma + \psi x), \tag{5.12}$$

where

$$\begin{aligned} x &= \rho \cos \phi, \\ y &= \rho \sin \phi, \\ \sigma &= \sqrt{u^2 + v^2}, \\ \tan \psi &= \frac{v}{u}. \end{aligned}$$

Equation (5.12) serves to raise any complex number to a complex power.

Curious consequences of Euler's formula (5.11) include that

$$\begin{aligned} e^{\pm i2\pi/4} &= \pm i, \\ e^{\pm i2\pi/2} &= -1, \\ e^{in2\pi} &= 1. \end{aligned} \tag{5.13}$$

For the natural logarithm of a complex number in light of Euler's formula, we have that

$$\ln w = \ln \left(\sigma e^{i\psi} \right) = \ln \sigma + i\psi. \tag{5.14}$$

more advanced mathematics, but briefer—constructs from Table 8.1 the Taylor series for $\exp i\phi$, $\cos \phi$ and $i \sin \phi$, then adds the latter two to show them equal to the first of the three. Such an alternate derivation lends little insight, perhaps, but at least it builds confidence that we actually knew what we were doing when we came up with the incredible (5.11).

5.5 Complex exponentials and de Moivre's theorem

Euler's formula (5.11) implies that complex numbers z_1 and z_2 can be written

$$\begin{aligned} z_1 &= \rho_1 e^{i\phi_1}, \\ z_2 &= \rho_2 e^{i\phi_2}. \end{aligned} \tag{5.15}$$

By the basic power properties of Table 2.2, then,

$$\begin{aligned} z_1 z_2 &= \rho_1 \rho_2 e^{i(\phi_1+\phi_2)} &&= \rho_1\rho_2 \exp[i(\phi_1+\phi_2)], \\ \frac{z_1}{z_2} &= \frac{\rho_1}{\rho_2} e^{i(\phi_1-\phi_2)} &&= \frac{\rho_1}{\rho_2} \exp[i(\phi_1-\phi_2)], \\ z^a &= \rho^a e^{ia\phi} &&= \rho^a \exp[ia\phi]. \end{aligned} \tag{5.16}$$

This is de Moivre's theorem, introduced in § 3.11.

5.6 Complex trigonometrics

Applying Euler's formula (5.11) to $+\phi$ then to $-\phi$, we have that

$$\begin{aligned} \exp(+i\phi) &= \cos\phi + i\sin\phi, \\ \exp(-i\phi) &= \cos\phi - i\sin\phi. \end{aligned}$$

Adding the two equations and solving for $\cos\phi$ yields

$$\cos\phi = \frac{\exp(+i\phi) + \exp(-i\phi)}{2}. \tag{5.17}$$

Subtracting the second equation from the first and solving for $\sin\phi$ yields

$$\sin\phi = \frac{\exp(+i\phi) - \exp(-i\phi)}{i2}. \tag{5.18}$$

Thus are the trigonometrics expressed in terms of complex exponentials.

The forms (5.17) and (5.18) suggest the definition of new functions

$$\cosh\phi \equiv \frac{\exp(+\phi) + \exp(-\phi)}{2}, \tag{5.19}$$

$$\sinh\phi \equiv \frac{\exp(+\phi) - \exp(-\phi)}{2}, \tag{5.20}$$

$$\tanh\phi \equiv \frac{\sinh\phi}{\cosh\phi}. \tag{5.21}$$

These are called the *hyperbolic functions.* Their inverses arccosh, etc., are defined in the obvious way. The Pythagorean theorem for trigonometrics (3.2) is that $\cos^2\phi + \sin^2\phi = 1$; and from (5.19) and (5.20) one can derive the hyperbolic analog:

$$\begin{aligned} \cos^2\phi + \sin^2\phi &= 1, \\ \cosh^2\phi - \sinh^2\phi &= 1. \end{aligned} \tag{5.22}$$

The notation $\exp i(\cdot)$ or $e^{i(\cdot)}$ is sometimes felt to be too bulky. Although less commonly seen than the other two, the notation

$$\operatorname{cis}(\cdot) \equiv \exp i(\cdot) = \cos(\cdot) + i\sin(\cdot)$$

is also conventionally recognized, as earlier seen in § 3.11. Also conventionally recognized are $\sin^{-1}(\cdot)$ and occasionally $\operatorname{asin}(\cdot)$ for $\arcsin(\cdot)$, and likewise for the several other trigs.

Replacing $z \leftarrow \phi$ in this section's several equations implies a coherent definition for trigonometric functions of a complex variable.

At this point in the development one begins to notice that the sin, cos, exp, cis, cosh and sinh functions are each really just different facets of the same mathematical phenomenon. Likewise their respective inverses: arcsin, arccos, ln, $-i\ln$, arccosh and arcsinh. Conventional names for these two mutually inverse families of functions are unknown to the author, but one might call them the *natural exponential* and *natural logarithmic families.* If the various tangent functions were included, one might call them the *trigonometric* and *inverse trigonometric families.*

5.7 Summary of properties

Table 5.1 gathers properties of the complex exponential from this chapter and from §§ 2.12, 3.11 and 4.4.

5.8 Derivatives of complex exponentials

This section computes the derivatives of the various trigonometric and inverse trigonometric functions.

5.8.1 Derivatives of sine and cosine

One can indeed compute derivatives of the sine and cosine functions from (5.17) and (5.18), but to do it in that way doesn't seem sporting. Better

Table 5.1: Complex exponential properties.

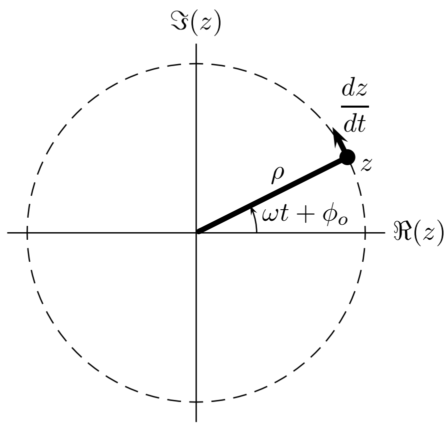
$$
\begin{aligned}
i^2 &= -1 = (-i)^2 \\
\frac{1}{i} &= -i \\
e^{i\phi} &= \cos\phi + i\sin\phi \\
e^{iz} &= \cos z + i\sin z \\
z_1 z_2 &= \rho_1\rho_2 e^{i(\phi_1+\phi_2)} = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \\
\frac{z_1}{z_2} &= \frac{\rho_1}{\rho_2} e^{i(\phi_1-\phi_2)} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2} \\
z^a &= \rho^a e^{ia\phi} \\
w^z &= e^{x\ln\sigma - \psi y} e^{i(y\ln\sigma + \psi x)} \\
\ln w &= \ln\sigma + i\psi
\end{aligned}
$$

$$
\begin{aligned}
\sin z &= \frac{e^{iz} - e^{-iz}}{i2} & \sinh z &= \frac{e^z - e^{-z}}{2} \\
\cos z &= \frac{e^{iz} + e^{-iz}}{2} & \cosh z &= \frac{e^z + e^{-z}}{2} \\
\tan z &= \frac{\sin z}{\cos z} & \tanh z &= \frac{\sinh z}{\cosh z}
\end{aligned}
$$

$$
\cos^2 z + \sin^2 z = 1 = \cosh^2 z - \sinh^2 z
$$

$$
\begin{aligned}
z &\equiv x + iy = \rho e^{i\phi} \\
w &\equiv u + iv = \sigma e^{i\psi} \\
\exp z &\equiv e^z \\
\operatorname{cis} z &\equiv \cos z + i\sin z = e^{iz} \\
\frac{d}{dz}\exp z &= \exp z \\
\frac{d}{dw}\ln w &= \frac{1}{w} \\
\frac{df/dz}{f(z)} &= \frac{d}{dz}\ln f(z)
\end{aligned}
$$

Figure 5.4: The derivatives of the sine and cosine functions.



applied style is to find the derivatives by observing directly the circle from which the sine and cosine functions come. Refer to Fig. 5.4. Suppose that the point z in the figure is not fixed but is traveling steadily about the circle such that

$$z(t) = (\rho)\left[\cos(\omega t + \phi_o) + i\sin(\omega t + \phi_o)\right]. \tag{5.23}$$

How fast then is the rate dz/dt, and in what Argand direction?

$$\frac{dz}{dt} = (\rho)\left[\frac{d}{dt}\cos(\omega t + \phi_o) + i\frac{d}{dt}\sin(\omega t + \phi_o)\right]. \tag{5.24}$$

Evidently,

- the speed is $|dz/dt| = (\rho)(d\phi/dt) = \rho\omega$;
- the direction is at right angles to the arm of ρ, which is to say that $\arg(dz/dt) = \phi + 2\pi/4$.

With these observations we can write that

$$\begin{aligned} \frac{dz}{dt} &= (\rho\omega)\left[\cos\left(\omega t + \phi_o + \frac{2\pi}{4}\right) + i\sin\left(\omega t + \phi_o + \frac{2\pi}{4}\right)\right] \\ &= (\rho\omega)\left[-\sin(\omega t + \phi_o) + i\cos(\omega t + \phi_o)\right]. \end{aligned} \tag{5.25}$$

Matching the real and imaginary parts of (5.24) against those of (5.25), we have that

$$\begin{aligned}\frac{d}{dt}\cos(\omega t+\phi_o) &= -\omega\sin(\omega t+\phi_o),\\ \frac{d}{dt}\sin(\omega t+\phi_o) &= +\omega\cos(\omega t+\phi_o).\end{aligned} \tag{5.26}$$

If $\omega = 1$ and $\phi_o = 0$, these are

$$\begin{aligned}\frac{d}{dt}\cos t &= -\sin t,\\ \frac{d}{dt}\sin t &= +\cos t.\end{aligned} \tag{5.27}$$

5.8.2 Derivatives of the trigonometrics

The derivatives of $\exp(\cdot)$, $\sin(\cdot)$ and $\cos(\cdot)$, eqns. (5.4) and (5.27) give. From these, with the help of (5.22) and of the derivative chain and product rules (§ 4.5), we can calculate[7] the derivatives in Table 5.2.

5.8.3 Derivatives of the inverse trigonometrics

Observe the pair

$$\begin{aligned}\frac{d}{dz}\exp z &= \exp z,\\ \frac{d}{dw}\ln w &= \frac{1}{w}.\end{aligned}$$

The natural exponential $\exp z$ belongs to the trigonometric family of functions, as does its derivative. The natural logarithm $\ln w$, by contrast, belongs to the inverse trigonometric family of functions, but its derivative is simpler, not a trigonometric or inverse trigonometric function at all. In Table 5.2, one notices that all the trigonometrics have trigonometric derivatives. By analogy with the natural logarithm, do all the inverse trigonometrics have simpler derivatives?

It turns out that they do. Refer to the account of the natural logarithm's derivative in § 5.2. Following a similar procedure, we have by successive steps

[7]The author has referred to [22, back endpaper] in checking some of these results.

Table 5.2: Derivatives of the trigonometrics.

$$\begin{aligned}
\frac{d}{dz}\exp z &= +\exp z & \frac{d}{dz}\frac{1}{\exp z} &= -\frac{1}{\exp z} \\
\frac{d}{dz}\sin z &= +\cos z & \frac{d}{dz}\frac{1}{\sin z} &= -\frac{1}{\tan z \sin z} \\
\frac{d}{dz}\cos z &= -\sin z & \frac{d}{dz}\frac{1}{\cos z} &= +\frac{\tan z}{\cos z}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\tan z &= +\left(1+\tan^2 z\right) &&= +\frac{1}{\cos^2 z} \\
\frac{d}{dz}\frac{1}{\tan z} &= -\left(1+\frac{1}{\tan^2 z}\right) &&= -\frac{1}{\sin^2 z}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\sinh z &= +\cosh z & \frac{d}{dz}\frac{1}{\sinh z} &= -\frac{1}{\tanh z \sinh z} \\
\frac{d}{dz}\cosh z &= +\sinh z & \frac{d}{dz}\frac{1}{\cosh z} &= -\frac{\tanh z}{\cosh z}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\tanh z &= 1-\tanh^2 z &&= +\frac{1}{\cosh^2 z} \\
\frac{d}{dz}\frac{1}{\tanh z} &= 1-\frac{1}{\tanh^2 z} &&= -\frac{1}{\sinh^2 z}
\end{aligned}$$

that

$$\begin{aligned}
\arcsin w &= z, \\
w &= \sin z, \\
\frac{dw}{dz} &= \cos z, \\
\frac{dw}{dz} &= \pm\sqrt{1-\sin^2 z}, \\
\frac{dw}{dz} &= \pm\sqrt{1-w^2}, \\
\frac{dz}{dw} &= \frac{\pm 1}{\sqrt{1-w^2}}, \\
\frac{d}{dw}\arcsin w &= \frac{\pm 1}{\sqrt{1-w^2}}.
\end{aligned} \tag{5.28}$$

Similarly,

$$\begin{aligned}
\arctan w &= z, \\
w &= \tan z, \\
\frac{dw}{dz} &= 1+\tan^2 z, \\
\frac{dw}{dz} &= 1+w^2, \\
\frac{dz}{dw} &= \frac{1}{1+w^2}, \\
\frac{d}{dw}\arctan w &= \frac{1}{1+w^2}.
\end{aligned} \tag{5.29}$$

Derivatives of the other inverse trigonometrics are found in the same way. Table 5.3 summarizes.

5.9 The actuality of complex quantities

Doing all this neat complex math, the applied mathematician can lose sight of some questions he probably ought to keep in mind: Is there really such a thing as a complex quantity in nature? If not, then hadn't we better avoid these complex quantities, leaving them to the professional mathematical theorists?

As developed by Oliver Heaviside in 1887 [2], the answer depends on your point of view. If I have 300 g of grapes and 100 g of grapes, then I

Table 5.3: Derivatives of the inverse trigonometrics.

$$
\begin{aligned}
\frac{d}{dw}\ln w &= \frac{1}{w} \\
\frac{d}{dw}\arcsin w &= \frac{\pm 1}{\sqrt{1-w^2}} \\
\frac{d}{dw}\arccos w &= \frac{\mp 1}{\sqrt{1-w^2}} \\
\frac{d}{dw}\arctan w &= \frac{1}{1+w^2} \\
\frac{d}{dw}\operatorname{arcsinh} w &= \frac{\pm 1}{\sqrt{w^2+1}} \\
\frac{d}{dw}\operatorname{arccosh} w &= \frac{\pm 1}{\sqrt{w^2-1}} \\
\frac{d}{dw}\operatorname{arctanh} w &= \frac{1}{1-w^2}
\end{aligned}
$$

have 400 g altogether. Alternately, if I have 500 g of grapes and -100 g of grapes, again I have 400 g altogether. (What does it mean to have -100 g of grapes? Maybe I ate some.) But what if I have $200 + i100$ g of grapes and $200 - i100$ g of grapes? Answer: again, 400 g.

Probably you would not choose to think of $200 + i100$ g of grapes and $200 - i100$ g of grapes, but because of (5.17) and (5.18), one often describes wave phenomena as linear superpositions (sums) of countervailing complex exponentials. Consider for instance the propagating wave

$$A\cos[\omega t - kz] = \frac{A}{2}\exp[+i(\omega t - kz)] + \frac{A}{2}\exp[-i(\omega t - kz)].$$

The benefit of splitting the real cosine into two complex parts is that while the magnitude of the cosine changes with time t, the magnitude of either exponential alone remains steady (see the circle in Fig. 5.3). It turns out to be much easier to analyze two complex wave quantities of constant magnitude than to analyze one real wave quantity of varying magnitude. Better yet, since each complex wave quantity is the complex conjugate of the other, the analyses thereof are mutually conjugate, too; so you normally needn't actually analyze the second. The one analysis suffices for both.[8] (It's like

[8]If the point is not immediately clear, an example: Suppose that by the Newton-

reflecting your sister's handwriting. To read her handwriting backward, you needn't ask her to try writing reverse with the wrong hand; you can just hold her regular script up to a mirror. Of course, this ignores the question of why one would want to reflect someone's handwriting in the first place; but anyway, reflecting—which is to say, conjugating—complex quantities often is useful.)

Some authors have gently denigrated the use of imaginary parts in physical applications as a mere mathematical trick, as though the parts were not actually there. Well, that is one way to treat the matter, but it is not the way this book recommends. Nothing in the mathematics *requires* you to regard the imaginary parts as physically nonexistent. You need not abuse Occam's razor! It is true by Euler's formula (5.11) that a complex exponential $\exp i\phi$ can be decomposed into a sum of trigonometrics. However, it is equally true by the complex trigonometric formulas (5.17) and (5.18) that *a trigonometric can be decomposed into a sum of complex exponentials.* So, if each can be decomposed into the other, then which of the two is the real decomposition? Answer: it depends on your point of view. Experience seems to recommend viewing the complex exponential as the basic element—as the element of which the trigonometrics are composed—rather than the other way around. From this point of view, it is (5.17) and (5.18) which are the real decomposition. Euler's formula itself is secondary.

The complex exponential method of offsetting imaginary parts offers an elegant yet practical mathematical way to model physical wave phenomena. So go ahead: regard the imaginary parts as actual. It doesn't hurt anything, and it helps with the math.

Raphson iteration (§ 4.8) you have found a root of the polynomial $x^3 + 2x^2 + 3x + 4$ at $x \approx -\mathrm{0x0.2D} + i\mathrm{0x1.8C}$. Where is there another root? Answer: at the complex conjugate, $x \approx -\mathrm{0x0.2D} - i\mathrm{0x1.8C}$. One need not actually run the Newton-Raphson again to find the conjugate root.

Chapter 6

Primes, roots and averages

This chapter gathers a few significant topics, each of whose treatment seems too brief for a chapter of its own.

6.1 Prime numbers

A *prime number*—or simply, a *prime*—is an integer greater than one, divisible only by one and itself. A *composite number* is an integer greater than one and not prime. A composite number can be composed as a product of two or more prime numbers. All positive integers greater than one are either composite or prime.

The mathematical study of prime numbers and their incidents constitutes *number theory,* and it is a deep area of mathematics. The deeper results of number theory seldom arise in applications,[1] however, so we shall confine our study of number theory in this book to one or two of its simplest, most broadly interesting results.

6.1.1 The infinite supply of primes

The first primes are evidently $2, 3, 5, 7, \mathrm{0xB}, \ldots$ Is there a last prime? To show that there is not, suppose that there were. More precisely, suppose that there existed exactly N primes, with N finite, letting $p_1, p_2, \ldots, p_N$ represent these primes from least to greatest. Now consider the product of

[1] The deeper results of number theory do arise in cryptography, or so the author has been led to understand. Although cryptography is literally an application of mathematics, its spirit is that of pure mathematics rather than of applied. If you seek cryptographic derivations, this book is probably not the one you want.

all the primes,

$$C = \prod_{k=1}^{N} p_k.$$

What of $C + 1$? Since $p_1 = 2$ divides C, it cannot divide $C + 1$. Similarly, since $p_2 = 3$ divides C, it also cannot divide $C + 1$. The same goes for $p_3 = 5$, $p_4 = 7$, $p_5 = \mathrm{0xB}$, etc. Apparently none of the primes in the p_k series divides $C + 1$, which implies either that $C + 1$ itself is prime, or that $C + 1$ is composed of primes not in the series. But the latter is assumed impossible on the ground that the p_k series includes all primes; and the former is assumed impossible on the ground that $C + 1 > C > p_N$, with p_N the greatest prime. The contradiction proves false the assumption which gave rise to it. The false assumption: that there were a last prime.

Thus there is no last prime. No matter how great a prime number one finds, a greater can always be found. The supply of primes is infinite.

Attributed to the ancient geometer Euclid, the foregoing proof is a classic example of mathematical *reductio ad absurdum,* or as usually styled in English, *proof by contradiction.*

(References: [20, Appendix 1]; [25]; [3, "Reductio ad absurdum," 02:36, 28 April 2006].)

6.1.2 Compositional uniqueness

Occasionally in mathematics, plausible assumptions can hide subtle logical flaws. One such plausible assumption is the assumption that every positive integer has a unique *prime factorization.* It is readily seen that the first several positive integers—$1 = ()$, $2 = (2^1)$, $3 = (3^1)$, $4 = (2^2)$, $5 = (5^1)$, $6 = (2^1)(3^1)$, $7 = (7^1)$, $8 = (2^3)$, ...—each have unique prime factorizations, but is this necessarily true of all positive integers?

To show that it is true, suppose that it were not.[2] More precisely, suppose that there did exist positive integers factorable each in two or more distinct ways, with the symbol C representing the least such integer. Noting that C must be composite (prime numbers by definition are each factorable

[2]Unfortunately the author knows no more elegant proof than this, yet cannot even cite this one properly. The author encountered the proof in some book over a decade ago. The identity of that book is now long forgotten.

only one way, like $5 = [5^1]$), let

$$
\begin{aligned}
C_p &\equiv \prod_{j=1}^{N_p} p_j, \\
C_q &\equiv \prod_{k=1}^{N_q} q_k, \\
C_p = C_q &= C, \\
p_j &\le p_{j+1}, \\
q_k &\le q_{k+1}, \\
p_1 &\le q_1, \\
N_p &> 1, \\
N_q &> 1,
\end{aligned}
$$

where C_p and C_q represent two distinct prime factorizations of the same number C and where the p_j and q_k are the respective primes ordered from least to greatest. We see that

$$p_j \neq q_k$$

for any j and k—that is, that the same prime cannot appear in both factorizations—because if the same prime r did appear in both then C/r would constitute an ambiguously factorable positive integer less than C, when we had already defined C to represent the least such. Among other effects, the fact that $p_j \neq q_k$ strengthens the definition $p_1 \le q_1$ to read

$$p_1 < q_1.$$

Let us now rewrite the two factorizations in the form

$$
\begin{aligned}
C_p &= p_1 A_p, \\
C_q &= q_1 A_q, \\
C_p = C_q &= C, \\
A_p &\equiv \prod_{j=2}^{N_p} p_j, \\
A_q &\equiv \prod_{k=2}^{N_q} q_k,
\end{aligned}
$$

where p_1 and q_1 are the least primes in their respective factorizations. Since C is composite and since $p_1 < q_1$, we have that

$$1 < p_1 < q_1 \le \sqrt{C} \le A_q < A_p < C,$$

which implies that

$$p_1 q_1 < C.$$

The last inequality lets us compose the new positive integer

$$B = C - p_1 q_1,$$

which might be prime or composite (or unity), but which either way enjoys a unique prime factorization because $B < C$, with C the least positive integer factorable two ways. Observing that some integer s which divides C necessarily also divides $C \pm ns$, we note that each of p_1 and q_1 necessarily divides B. This means that B's unique factorization includes both p_1 and q_1, which further means that the product $p_1 q_1$ divides B. But if $p_1 q_1$ divides B, then it divides $B + p_1 q_1 = C$, also.

Let E represent the positive integer which results from dividing C by $p_1 q_1$:

$$E \equiv \frac{C}{p_1 q_1}.$$

Then,

$$\begin{aligned} Eq_1 &= \frac{C}{p_1} = A_p, \\ Ep_1 &= \frac{C}{q_1} = A_q. \end{aligned}$$

That $Eq_1 = A_p$ says that q_1 divides A_p. But $A_p < C$, so A_p's prime factorization is unique—and we see above that A_p's factorization *does not include any* q_k, not even q_1. The contradiction proves false the assumption which gave rise to it. The false assumption: that there existed a least composite number C prime-factorable in two distinct ways.

Thus no positive integer is ambiguously factorable. Prime factorizations are always unique.

We have observed at the start of this subsection that plausible assumptions can hide subtle logical flaws. Indeed this is so. Interestingly however, the plausible assumption of the present subsection has turned out absolutely correct; we have just had to do some extra work to prove it. Such effects are typical on the shadowed frontier where applied shades into pure mathematics: with sufficient experience and with a firm grasp of the model at hand, if you think that it's true, then it probably is. Judging when to delve into the mathematics anyway, seeking a more rigorous demonstration of a proposition one feels pretty sure is correct, is a matter of applied mathematical style. It depends on how sure one feels, and more importantly on whether the unsureness felt is true uncertainty or is just an unaccountable desire for more precise mathematical definition (if the latter, then unlike the author you may have the right temperament to become a professional mathematician). The author does judge the present subsection's proof to be worth the applied effort; but nevertheless, when one lets logical minutiae distract him to too great a degree, he admittedly begins to drift out of the applied mathematical realm that is the subject of this book.

6.1.3 Rational and irrational numbers

A *rational number* is a finite real number expressible as a ratio of integers

$$x = \frac{p}{q}, \quad q \neq 0.$$

The ratio is *fully reduced* if p and q have no prime factors in common. For instance, 4/6 is not fully reduced, whereas 2/3 is.

An *irrational number* is a finite real number which is not rational. For example, $\sqrt{2}$ is irrational. In fact any $x = \sqrt{n}$ is irrational unless integral; there is no such thing as a $\sqrt{n}$ which is not an integer but is rational.

To prove[3] the last point, suppose that there did exist a fully reduced

$$x = \frac{p}{q} = \sqrt{n}, \quad p > 0, \ q > 1,$$

where p, q and n are all integers. Squaring the equation, we have that

$$\frac{p^2}{q^2} = n,$$

which form is evidently also fully reduced. But if $q > 1$, then the fully reduced $n = p^2/q^2$ is not an integer as we had assumed that it was. The contradiction proves false the assumption which gave rise to it. Hence there exists no rational, nonintegral $\sqrt{n}$, as was to be demonstrated. The proof is readily extended to show that any $x = n^{j/k}$ is irrational if nonintegral, the

[3] A proof somewhat like the one presented here is found in [20, Appendix 1].

extension by writing $p^k/q^k = n^j$ then following similar steps as those this paragraph outlines.

That's all the number theory the book treats; but in applied math, so little will take you pretty far. Now onward we go to other topics.

6.2 The existence and number of polynomial roots

This section shows that an Nth-order polynomial must have exactly N roots.

6.2.1 Polynomial roots

Consider the quotient $B(z)/A(z)$, where

$$\begin{aligned} A(z) &= z - \alpha, \\ B(z) &= \sum_{k=0}^{N} b_k z^k, \quad N > 0,\ b_N \neq 0, \\ B(\alpha) &= 0. \end{aligned}$$

In the long-division symbology of Table 2.3,

$$B(z) = A(z)Q_0(z) + R_0(z),$$

where $Q_0(z)$ is the quotient and $R_0(z)$, a remainder. In this case the divisor $A(z) = z - \alpha$ has first order, and as § 2.6.2 has observed, first-order divisors leave zeroth-order, constant remainders $R_0(z) = \rho$. Thus substituting leaves

$$B(z) = (z - \alpha)Q_0(z) + \rho.$$

When $z = \alpha$, this reduces to

$$B(\alpha) = \rho.$$

But $B(\alpha) = 0$, so

$$\rho = 0.$$

Evidently the division leaves no remainder ρ, which is to say that *$z - \alpha$ exactly divides every polynomial $B(z)$ of which $z = \alpha$ is a root.*

Note that if the polynomial $B(z)$ has order N, then the quotient $Q(z) = B(z)/(z - \alpha)$ has exactly order $N - 1$. That is, the leading, z^{N-1} term of the quotient is never null. The reason is that if the leading term were null, if $Q(z)$ had order less than $N - 1$, then $B(z) = (z - \alpha)Q(z)$ could not possibly have order N as we have assumed.

6.2.2 The fundamental theorem of algebra

The *fundamental theorem of algebra* holds that any polynomial $B(z)$ of order N can be factored

$$B(z) = \sum_{k=0}^{N} b_k z^k = b_N \prod_{k=1}^{N} (z - \alpha_k), \quad b_N \neq 0, \tag{6.1}$$

where the α_k are the N roots of the polynomial.[4]

To prove the theorem, it suffices to show that all polynomials of order $N > 0$ have at least one root; for if a polynomial of order N has a root α_N, then according to § 6.2.1 one can divide the polynomial by $z - \alpha_N$ to obtain a new polynomial of order $N - 1$. To the new polynomial the same logic applies; if it has at least one root α_{N-1}, then one can divide by $z - \alpha_{N-1}$ to obtain yet another polynomial of order $N - 2$; and so on, one root extracted at each step, factoring the polynomial step by step into the desired form $b_N \prod_{k=1}^{N} (z - \alpha_k)$.

It remains however to show that there exists no polynomial $B(z)$ of order $N > 0$ lacking roots altogether. To show that there is no such polynomial, consider the locus[5] of all $B(\rho e^{i\phi})$ in the Argand range plane (Fig. 2.5), where $z = \rho e^{i\phi}$, ρ is held constant, and ϕ is variable. Because $e^{i(\phi+n2\pi)} = e^{i\phi}$ and no fractional powers of z appear in (6.1), this locus forms a closed loop. At very large ρ, the $b_N z^N$ term dominates $B(z)$, so the locus there evidently has the general character of $b_N \rho^N e^{iN\phi}$. As such, the locus is nearly but not quite a circle at radius $b_N \rho^N$ from the Argand origin $B(z) = 0$, revolving N times at that great distance before exactly repeating. On the other hand, when $\rho = 0$ the entire locus collapses on the single point $B(0) = b_0$.

Now consider the locus at very large ρ again, but this time let ρ slowly shrink. Watch the locus as ρ shrinks. The locus is like a great string or rubber band, joined at the ends and looped in N great loops. As ρ shrinks smoothly, the string's shape changes smoothly. Eventually ρ disappears and the entire string collapses on the point $B(0) = b_0$. Since the string originally has looped N times at great distance about the Argand origin, but at the end has collapsed on a single point, then at some time between it must have swept through the origin and every other point within the original loops.

[4]Professional mathematicians typically state the theorem in a slightly different form. They also prove it in rather a different way. [15, Ch. 10, Prob. 74]

[5]A *locus* is the geometric collection of points which satisfy a given criterion. For example, the locus of all points in a plane at distance ρ from a point O is a circle; the locus of all points in three-dimensional space equidistant from two points P and Q is a plane; etc.

After all, $B(z)$ is everywhere differentiable, so the string can only *sweep* as ρ decreases; it can never skip. The Argand origin lies inside the loops at the start but outside at the end. If so, then the values of ρ and ϕ precisely where the string has swept through the origin by definition constitute a root $B(\rho e^{i\phi}) = 0$. Thus as we were required to show, $B(z)$ does have at least one root, which observation completes the applied demonstration of the fundamental theorem of algebra.

The fact that the roots exist is one thing. Actually finding the roots numerically is another matter. For a quadratic (second order) polynomial, (2.2) gives the roots. For cubic (third order) and quartic (fourth order) polynomials, formulas for the roots are known (see Ch. 10) though seemingly not so for quintic (fifth order) and higher-order polynomials;[6] but the Newton-Raphson iteration (§ 4.8) can be used to locate a root numerically in any case. The Newton-Raphson is used to extract one root (*any* root) at each step as described above, reducing the polynomial step by step until all the roots are found.

The reverse problem, finding the polynomial given the roots, is much easier: one just multiplies out $\prod_k (z - \alpha_k)$, as in (6.1).

6.3 Addition and averages

This section discusses the two basic ways to add numbers and the three basic ways to calculate averages of them.

6.3.1 Serial and parallel addition

Consider the following problem. There are three masons. The strongest and most experienced of the three, Adam, lays 120 bricks per hour.[7] Next is Brian who lays 90. Charles is new; he lays only 60. Given eight hours, how many bricks can the three men lay? Answer:

$$(8 \text{ hours})(120 + 90 + 60 \text{ bricks per hour}) = 2160 \text{ bricks}.$$

Now suppose that we are told that Adam can lay a brick every 30 seconds; Brian, every 40 seconds; Charles, every 60 seconds. How much time do the

[6] In a celebrated theorem of pure mathematics [27, "Abel's impossibility theorem"], it is said to be shown that no such formula even exists, given that the formula be constructed according to certain rules. Undoubtedly the theorem is interesting to the professional mathematician, but to the applied mathematician it probably suffices to observe merely that no such formula is known.

[7] The figures in the example are in decimal notation.

three men need to lay 2160 bricks? Answer:

$$\begin{aligned}\frac{2160 \text{ bricks}}{\frac{1}{30}+\frac{1}{40}+\frac{1}{60} \text{ bricks per second}} &= 28{,}800 \text{ seconds}\left(\frac{1 \text{ hour}}{3600 \text{ seconds}}\right) \\ &= 8 \text{ hours.}\end{aligned}$$

The two problems are precisely equivalent. Neither is stated in simpler terms than the other. The notation used to solve the second is less elegant, but fortunately there exists a better notation:

$$(2160 \text{ bricks})(30 \parallel 40 \parallel 60 \text{ seconds per brick}) = 8 \text{ hours},$$

where

$$\frac{1}{30 \parallel 40 \parallel 60} = \frac{1}{30} + \frac{1}{40} + \frac{1}{60}.$$

The operator $\parallel$ is called the *parallel addition* operator. It works according to the law

$$\frac{1}{a \parallel b} = \frac{1}{a} + \frac{1}{b}, \tag{6.2}$$

where the familiar operator $+$ is verbally distinguished from the $\parallel$ when necessary by calling the $+$ the *serial addition* or *series addition* operator. With (6.2) and a bit of arithmetic, the several parallel-addition identities of Table 6.1 are soon derived.

The writer knows of no conventional notation for parallel sums of series, but suggests that the notation which appears in the table,

$$\sum_{k=a}^{b} \parallel f(k) \equiv f(a) \parallel f(a+1) \parallel f(a+2) \parallel \cdots \parallel f(b),$$

might serve if needed.

Assuming that none of the values involved is negative, one can readily show that[8]

$$a \parallel x \le b \parallel x \quad \text{iff} \quad a \le b. \tag{6.3}$$

This is intuitive. Counterintuitive, perhaps, is that

$$a \parallel x \le a. \tag{6.4}$$

Because we have all learned as children to count in the sensible manner $1, 2, 3, 4, 5, \ldots$—rather than as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$—serial addition (+) seems

[8]The word *iff* means, "if and only if."

Table 6.1: Parallel and serial addition identities.

$$
\begin{array}{rclcrcl}
\dfrac{1}{a \,\|\, b} & = & \dfrac{1}{a} + \dfrac{1}{b} & \qquad & \dfrac{1}{a + b} & = & \dfrac{1}{a} \,\|\, \dfrac{1}{b} \\
a \,\|\, b & = & \dfrac{ab}{a + b} & & a + b & = & \dfrac{ab}{a \,\|\, b} \\
a \,\|\, \dfrac{1}{b} & = & \dfrac{a}{1 + ab} & & a + \dfrac{1}{b} & = & \dfrac{a}{1 \,\|\, ab} \\
a \,\|\, b & = & b \,\|\, a & & a + b & = & b + a \\
a \,\|\, (b \,\|\, c) & = & (a \,\|\, b) \,\|\, c & & a + (b + c) & = & (a + b) + c \\
a \,\|\, \infty = \infty \,\|\, a & = & a & & a + 0 = 0 + a & = & a \\
a \,\|\, (-a) & = & \infty & & a + (-a) & = & 0 \\
(a)(b \,\|\, c) & = & ab \,\|\, ac & & (a)(b + c) & = & ab + ac \\
\dfrac{1}{\sum_k \,\|\, a_k} & = & \displaystyle\sum_k \frac{1}{a_k} & & \dfrac{1}{\sum_k a_k} & = & \displaystyle\sum_k \,\|\, \frac{1}{a_k}
\end{array}
$$

more natural than parallel addition ($\|$) does. The psychological barrier is hard to breach, yet for many purposes parallel addition is in fact no less fundamental. Its rules are inherently neither more nor less complicated, as Table 6.1 illustrates; yet outside the electrical engineering literature the parallel addition notation is seldom seen.[9] Now that you have seen it, you can use it. There is profit in learning to think both ways. (Exercise: counting from zero serially goes $0, 1, 2, 3, 4, 5, \ldots$; how does the parallel analog go?)

Convention has no special symbol for parallel subtraction, incidentally. One merely writes

$$a \,\|\, (-b),$$

which means exactly what it appears to mean.

(Reference: [21, eqn. 1.27].)

6.3.2 Averages

Let us return to the problem of the preceding section. Among the three masons, what is their average productivity? The answer depends on how you look at it. On the one hand,

$$\frac{120 + 90 + 60 \text{ bricks per hour}}{3} = 90 \text{ bricks per hour.}$$

On the other hand,

$$\frac{30 + 40 + 60 \text{ seconds per brick}}{3} = 43\tfrac{1}{3} \text{ seconds per brick.}$$

These two figures are not the same. That is, $1/(43\frac{1}{3}$ seconds per brick$) \neq$ 90 bricks per hour. Yet both figures are valid. Which figure you choose depends on what you want to calculate. A common mathematical error among businesspeople is not to realize that both averages are possible and that they yield different numbers (if the businessperson quotes in bricks per hour, the productivities average one way; if in seconds per brick, the other way; yet some businesspeople will never clearly consider the difference). Realizing this, the clever businessperson might negotiate a contract so that the average used worked to his own advantage.[10]

[9] In electric circuits, loads are connected in parallel as often as, in fact probably more often than, they are connected in series. Parallel addition gives the electrical engineer a neat way of adding the impedances of parallel-connected loads.

[10] "And what does the author know about business?" comes the rejoinder.

When it is unclear which of the two averages is more appropriate, a third average is available, the *geometric mean*

$$[(120)(90)(60)]^{1/3} \text{ bricks per hour.}$$

The geometric mean does not have the problem either of the two averages discussed above has. The inverse geometric mean

$$[(30)(40)(60)]^{1/3} \text{ seconds per brick}$$

implies the same average productivity. The mathematically savvy sometimes prefer the geometric mean over either of the others for this reason.

Generally, the *arithmetic, geometric* and *harmonic means* are defined

$$\begin{aligned}
\mu &\equiv \frac{\sum_k w_k x_k}{\sum_k w_k} = \left(\sum_k \| \frac{1}{w_k}\right)\left(\sum_k w_k x_k\right), &(6.5)\\
\mu_\Pi &\equiv \left[\prod_k x_k^{w_k}\right]^{1/\sum_k w_k} = \left[\prod_k x_k^{w_k}\right]^{\sum_k \| 1/w_k}, &(6.6)\\
\mu_\| &\equiv \frac{\sum_k \| x_k/w_k}{\sum_k \| 1/w_k} = \left(\sum_k w_k\right)\left(\sum_k \| \frac{x_k}{w_k}\right), &(6.7)
\end{aligned}$$

where the x_k are the several samples and the w_k are weights. For two samples weighted equally, these are

$$\begin{aligned}
\mu &= \frac{a+b}{2}, &(6.8)\\
\mu_\Pi &= \sqrt{ab}, &(6.9)\\
\mu_\| &= 2(a \| b). &(6.10)
\end{aligned}$$

The rejoinder is fair enough. If the author wanted to demonstrate his business acumen (or lack thereof) he'd do so elsewhere not here! There are a lot of good business books out there and this is not one of them.

The fact remains nevertheless that businesspeople sometimes use mathematics in peculiar ways, making relatively easy problems harder and more mysterious than the problems need to be. If you have ever encountered the little monstrosity of an approximation banks (at least in the author's country) actually use in place of (9.12) to accrue interest and amortize loans, then you have met the difficulty.

Trying to convince businesspeople that their math is wrong, incidentally, is in the author's experience usually a waste of time. Some businesspeople are mathematically rather sharp—as you presumably are if you are in business and are reading these words—but as for most: when real mathematical ability is needed, that's what they hire engineers, architects and the like for. The author is not sure, but somehow he doubts that many boards of directors would be willing to bet the company on a financial formula containing some mysterious-looking e^x. Business demands other talents.

If $a \geq 0$ and $b \geq 0$, then by successive steps,[11]

$$
\begin{aligned}
0 &\leq (a-b)^2, \\
0 &\leq a^2 - 2ab + b^2, \\
4ab &\leq a^2 + 2ab + b^2, \\
2\sqrt{ab} &\leq a + b, \\
\frac{2\sqrt{ab}}{a+b} &\leq \quad 1 \quad \leq \frac{a+b}{2\sqrt{ab}}, \\
\frac{2ab}{a+b} &\leq \sqrt{ab} \leq \frac{a+b}{2}, \\
2(a \parallel b) &\leq \sqrt{ab} \leq \frac{a+b}{2}.
\end{aligned}
$$

That is,

$$\mu_{\parallel} \leq \mu_{\Pi} \leq \mu. \tag{6.11}$$

The arithmetic mean is greatest and the harmonic mean, least; with the geometric mean falling between.

Does (6.11) hold when there are several nonnegative samples of various nonnegative weights? To show that it does, consider the case of $N = 2^m$ nonnegative samples of equal weight. Nothing prevents one from dividing such a set of samples in half, considering each subset separately, for if (6.11) holds for each subset individually then surely it holds for the whole set (this is because the average of the whole set is itself the *average of the two subset averages,* where the word "average" signifies the arithmetic, geometric or harmonic mean as appropriate). But each subset can further be divided in half, then each subsubset can be divided in half again, and so on until each smallest group has two members only—in which case we already know that (6.11) obtains. Starting there and recursing back, we have that (6.11)

[11] The steps are logical enough, but the motivation behind them remains inscrutable until the reader realizes that the writer originally worked the steps out backward with his pencil, from the last step to the first. Only then did he reverse the order and write the steps formally here. The writer had no idea that he was supposed to start from $0 \leq (a-b)^2$ until his pencil working backward showed him. "Begin with the end in mind," the saying goes. In this case the saying is right.

The same reading strategy often clarifies inscrutable math. When you can follow the logic but cannot understand what could possibly have inspired the writer to conceive the logic in the first place, try reading backward.

obtains for the entire set. Now consider that a sample of any weight can be approximated arbitrarily closely by several samples of weight $1/2^m$, provided that m is sufficiently large. By this reasoning, (6.11) holds for any nonnegative weights of nonnegative samples, which was to be demonstrated.

Chapter 7

The integral

Chapter 4 has observed that the mathematics of calculus concerns a complementary pair of questions:

- Given some function $f(t)$, what is the function's instantaneous rate of change, or *derivative,* $f'(t)$?
- Interpreting some function $f'(t)$ as an instantaneous rate of change, what is the corresponding accretion, or *integral,* $f(t)$?

Chapter 4 has built toward a basic understanding of the first question. This chapter builds toward a basic understanding of the second. The understanding of the second question constitutes the concept of the integral, one of the profoundest ideas in all of mathematics.

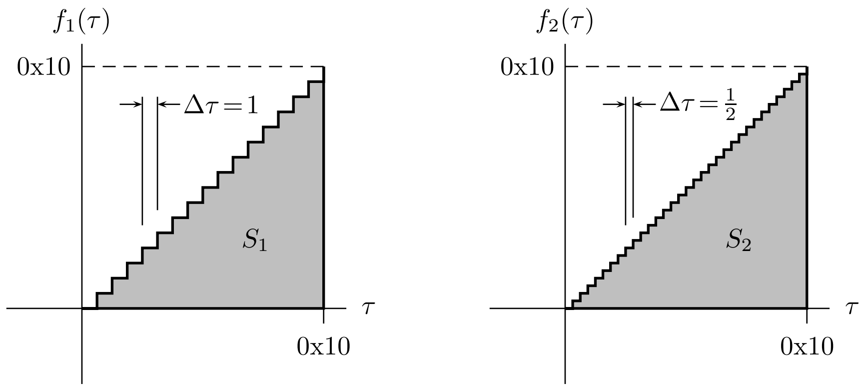
This chapter, which introduces the integral, is undeniably a hard chapter.

Experience knows no reliable way to teach the integral adequately to the uninitiated except through dozens or hundreds of pages of suitable examples and exercises, yet the book you are reading cannot be that kind of book. The sections of the present chapter concisely treat matters which elsewhere rightly command chapters or whole books of their own. Concision can be a virtue—and by design, nothing essential is omitted here—but the bold novice who wishes to learn the integral from these pages alone faces a daunting challenge. It can be done. However, for less intrepid readers who quite reasonably prefer a gentler initiation, [14] is warmly recommended.

7.1 The concept of the integral

An *integral* is a finite accretion or sum of an infinite number of infinitesimal elements. This section introduces the concept.

Figure 7.1: Areas representing discrete sums.



7.1.1 An introductory example

Consider the sums

$$
\begin{aligned}
S_1 &= \sum_{k=0}^{\mathrm{0x10}-1} k, \\
S_2 &= \frac{1}{2}\sum_{k=0}^{\mathrm{0x20}-1} \frac{k}{2}, \\
S_4 &= \frac{1}{4}\sum_{k=0}^{\mathrm{0x40}-1} \frac{k}{4}, \\
S_8 &= \frac{1}{8}\sum_{k=0}^{\mathrm{0x80}-1} \frac{k}{8}, \\
&\;\;\vdots \\
S_n &= \frac{1}{n}\sum_{k=0}^{(\mathrm{0x10})n-1} \frac{k}{n}.
\end{aligned}
$$

What do these sums represent? One way to think of them is in terms of the shaded areas of Fig. 7.1. In the figure, S_1 is composed of several tall, thin rectangles of width 1 and height k; S_2, of rectangles of width 1/2 and height

$k/2$.[1] As n grows, the shaded region in the figure looks more and more like a triangle of base length $b = \mathrm{0x10}$ and height $h = \mathrm{0x10}$. In fact it appears that

$$\lim_{n\to\infty} S_n = \frac{bh}{2} = \mathrm{0x80},$$

or more tersely

$$S_\infty = \mathrm{0x80},$$

is the area the increasingly fine stairsteps approach.

Notice how we have evaluated S_∞, the sum of an infinite number of infinitely narrow rectangles, without actually adding anything up. We have taken a shortcut directly to the total.

In the equation

$$S_n = \frac{1}{n} \sum_{k=0}^{(\mathrm{0x10})n-1} \frac{k}{n},$$

let us now change the variables

$$\begin{aligned} \tau &\leftarrow \frac{k}{n}, \\ \Delta\tau &\leftarrow \frac{1}{n}, \end{aligned}$$

to obtain the representation

$$S_n = \Delta\tau \sum_{k=0}^{(\mathrm{0x10})n-1} \tau;$$

or more properly,

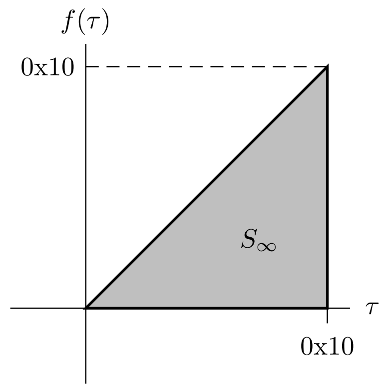
$$S_n = \sum_{k=0}^{(k|_{\tau=\mathrm{0x10}})-1} \tau\,\Delta\tau,$$

where the notation $k|_{\tau=\mathrm{0x10}}$ indicates the value of k when $\tau = \mathrm{0x10}$. Then

$$S_\infty = \lim_{\Delta\tau\to 0^+} \sum_{k=0}^{(k|_{\tau=\mathrm{0x10}})-1} \tau\,\Delta\tau,$$

[1] If the reader does not fully understand this paragraph's illustration, if the relation of the sum to the area seems unclear, the reader is urged to pause and consider the illustration carefully until he does understand it. If it still seems unclear, then the reader should probably suspend reading here and go study a good basic calculus text like [14]. The concept is important.

Figure 7.2: An area representing an infinite sum of infinitesimals. (Observe that the infinitesimal $d\tau$ is now too narrow to show on this scale. Compare against $\Delta\tau$ in Fig. 7.1.)



in which it is conventional as $\Delta\tau$ vanishes to change the symbol $d\tau \leftarrow \Delta\tau$, where $d\tau$ is the infinitesimal of Ch. 4:

$$S_\infty = \lim_{d\tau\to 0^+} \sum_{k=0}^{(k|_{\tau=0\text{x}10})-1} \tau\, d\tau.$$

The symbol $\lim_{d\tau\to 0^+} \sum_{k=0}^{(k|_{\tau=0\text{x}10})-1}$ is cumbersome, so we replace it with the new symbol[2] $\int_0^{0\text{x}10}$ to obtain the form

$$S_\infty = \int_0^{0\text{x}10} \tau\, d\tau.$$

This means, "stepping in infinitesimal intervals of $d\tau$, the sum of all $\tau\, d\tau$ from $\tau = 0$ to $\tau = 0\text{x}10$." Graphically, it is the shaded area of Fig. 7.2.

[2]Like the Greek S, $\sum$, denoting discrete summation, the seventeenth century-styled Roman S, $\int$, stands for Latin "summa," English "sum." See [3, "Long s," 14:54, 7 April 2006].

7.1.2 Generalizing the introductory example

Now consider a generalization of the example of § 7.1.1:

$$S_n = \frac{1}{n} \sum_{k=an}^{bn-1} f\left(\frac{k}{n}\right).$$

(In the example of § 7.1.1, $f(\tau)$ was the simple $f(\tau) = \tau$, but in general it could be any function.) With the change of variables

$$\begin{aligned} \tau &\leftarrow \frac{k}{n}, \\ \Delta\tau &\leftarrow \frac{1}{n}, \end{aligned}$$

this is

$$S_n = \sum_{k=(k|_{\tau=a})}^{(k|_{\tau=b})-1} f(\tau)\,\Delta\tau.$$

In the limit,

$$S_\infty = \lim_{d\tau\to 0^+} \sum_{k=(k|_{\tau=a})}^{(k|_{\tau=b})-1} f(\tau)\,d\tau = \int_a^b f(\tau)\,d\tau.$$

This is the *integral* of $f(\tau)$ in the interval $a < \tau < b$. It represents the area under the curve of $f(\tau)$ in that interval.

7.1.3 The balanced definition and the trapezoid rule

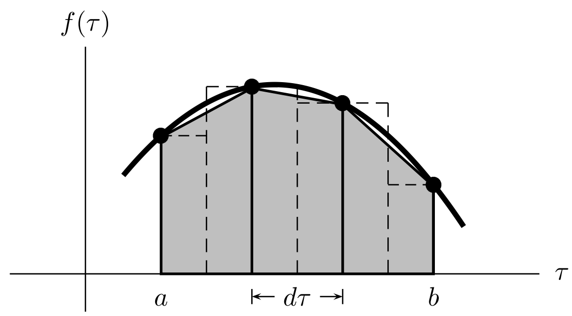
Actually, just as we have defined the derivative in the balanced form (4.15), we do well to define the integral in balanced form, too:

$$\int_a^b f(\tau)\,d\tau \equiv \lim_{d\tau\to 0^+} \left\{ \frac{f(a)\,d\tau}{2} + \sum_{k=(k|_{\tau=a})+1}^{(k|_{\tau=b})-1} f(\tau)\,d\tau + \frac{f(b)\,d\tau}{2} \right\}. \tag{7.1}$$

Here, the first and last integration samples are each balanced "on the edge," half within the integration domain and half without.

Equation (7.1) is known as the *trapezoid rule.* Figure 7.3 depicts it. The name "trapezoid" comes of the shapes of the shaded integration elements in the figure. Observe however that it makes no difference whether one regards the shaded trapezoids or the dashed rectangles as the actual integration

Figure 7.3: Integration by the trapezoid rule (7.1). Notice that the shaded and dashed areas total the same.



elements; the total integration area is the same either way.[3] The important point to understand is that the integral is conceptually just a sum. It is a sum of an infinite number of infinitesimal elements as $d\tau$ tends to vanish, but a sum nevertheless; nothing more.

Nothing actually requires the integration element width $d\tau$ to remain constant from element to element, incidentally. Constant widths are usually easiest to handle but variable widths find use in some cases. The only requirement is that $d\tau$ remain infinitesimal. (For further discussion of the point, refer to the treatment of the Leibnitz notation in § 4.4.2.)

7.2 The antiderivative and the fundamental theorem of calculus

If

$$S(x) \equiv \int_a^x g(\tau)\,d\tau,$$

[3]The trapezoid rule (7.1) is perhaps the most straightforward, general, robust way to define the integral, but other schemes are possible, too. For example, one can give the integration elements quadratically curved tops which more nearly track the actual curve. That scheme is called *Simpson's rule.* [A section on Simpson's rule might be added to the book at some later date.]

then what is the derivative dS/dx? After some reflection, one sees that the derivative must be

$$\frac{dS}{dx} = g(x).$$

This is because the action of the integral is to compile or accrete the area under a curve. The integral accretes area at a rate proportional to the curve's height $f(\tau)$: the higher the curve, the faster the accretion. In this way one sees that the integral and the derivative are inverse operators; the one inverts the other. The integral is the *antiderivative.*

More precisely,

$$\int_a^b \frac{df}{d\tau}\,d\tau = f(\tau)|_a^b, \tag{7.2}$$

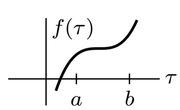
where the notation $f(\tau)|_a^b$ or $[f(\tau)]_a^b$ means $f(b) - f(a)$.

The importance of (7.2), fittingly named the *fundamental theorem of calculus* [14, § 11.6][22, § 5-4][3, "Fundamental theorem of calculus," 06:29, 23 May 2006], can hardly be overstated. As the formula which ties together the complementary pair of questions asked at the chapter's start, (7.2) is of utmost importance in the practice of mathematics. The idea behind the formula is indeed simple once grasped, but to grasp the idea firmly in the first place is not entirely trivial.[4] The idea is simple but big. The reader

[4]Having read from several calculus books and, like millions of others perhaps including the reader, having sat years ago in various renditions of the introductory calculus lectures in school, the author has never yet met a more convincing demonstration of (7.2) than the formula itself. Somehow the underlying idea is too simple, too profound to explain. It's like trying to explain how to drink water, or how to count or to add. Elaborate explanations and their attendant constructs and formalities are indeed possible to contrive, but the idea itself is so simple that somehow such contrivances seem to obscure the idea more than to reveal it.

One ponders the formula (7.2) a while, then the idea dawns on him.

If you want some help pondering, try this: Sketch some arbitrary function $f(\tau)$ on a set of axes at the bottom of a piece of paper—some squiggle of a curve like



will do nicely—then on a separate set of axes directly above the first, sketch the corresponding slope function $df/d\tau$. Mark two points a and b on the common horizontal axis; then on the upper, $df/d\tau$ plot, shade the integration area under the curve. Now consider (7.2) in light of your sketch.

There. Does the idea not dawn?

Another way to see the truth of the formula begins by canceling its $(1/d\tau)\,d\tau$ to obtain the form $\int_{\tau=a}^b df = f(\tau)|_a^b$. If this way works better for you, fine; but make sure that you understand it the other way, too.

Table 7.1: Basic derivatives for the antiderivative.

$$\int_a^b \frac{df}{d\tau}\,d\tau = f(\tau)|_a^b$$

$$\begin{aligned}
\tau^{a-1} &= \frac{d}{d\tau}\left(\frac{\tau^a}{a}\right), & a &\neq 0 \\
\frac{1}{\tau} &= \frac{d}{d\tau}\ln\tau, & \ln 1 &= 0 \\
\exp\tau &= \frac{d}{d\tau}\exp\tau, & \exp 0 &= 1 \\
\cos\tau &= \frac{d}{d\tau}\sin\tau, & \sin 0 &= 0 \\
\sin\tau &= \frac{d}{d\tau}\left(-\cos\tau\right), & \cos 0 &= 1
\end{aligned}$$

is urged to pause now and ponder the formula thoroughly until he feels reasonably confident that indeed he does grasp it and the important idea it represents. One is unlikely to do much higher mathematics without this formula.

As an example of the formula's use, consider that because $(d/d\tau)(\tau^3/6) = \tau^2/2$, it follows that

$$\int_2^x \frac{\tau^2\,d\tau}{2} = \int_2^x \frac{d}{d\tau}\left(\frac{\tau^3}{6}\right)d\tau = \left.\frac{\tau^3}{6}\right|_2^x = \frac{x^3-8}{6}.$$

Gathering elements from (4.21) and from Tables 5.2 and 5.3, Table 7.1 lists a handful of the simplest, most useful derivatives for antiderivative use. Section 9.1 speaks further of the antiderivative.

7.3 Operators, linearity and multiple integrals

This section presents the operator concept, discusses linearity and its consequences, treats the transitivity of the summational and integrodifferential operators, and introduces the multiple integral.

7.3.1 Operators

An *operator* is a mathematical agent which combines several values of a function.

Such a definition, unfortunately, is extraordinarily unilluminating to those who do not already know what it means. A better way to introduce the operator is by giving examples. Operators include $+$, $-$, multiplication, division, $\sum$, $\prod$, $\int$ and ∂. The essential action of an operator is to take several values of a function and combine them in some way. For example, $\prod$ is an operator in

$$\prod_{k=1}^{5}(2k-1) = (1)(3)(5)(7)(9) = \mathrm{0x3B1}.$$

Notice that the operator has acted to remove the variable k from the expression $2k - 1$. The k appears on the equation's left side but not on its right. The operator has used the variable up. Such a variable, used up by an operator, is a *dummy variable,* as encountered earlier in § 2.3.

7.3.2 A formalism

But then how are $+$ and $-$ operators? They don't use any dummy variables up, do they? Well, it depends on how you look at it. Consider the sum $S = 3 + 5$. One can write this as

$$S = \sum_{k=0}^{1} f(k),$$

where

$$f(k) \equiv \begin{cases} 3 & \text{if } k = 0, \\ 5 & \text{if } k = 1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then,

$$S = \sum_{k=0}^{1} f(k) = f(0) + f(1) = 3 + 5 = 8.$$

By such admittedly excessive formalism, the $+$ operator can indeed be said to use a dummy variable up. The point is that $+$ is in fact an operator just like the others.

Another example of the kind:

$$\begin{aligned} D &= g(z) - h(z) + p(z) + q(z) \\ &= g(z) - h(z) + p(z) - 0 + q(z) \\ &= \Phi(0,z) - \Phi(1,z) + \Phi(2,z) - \Phi(3,z) + \Phi(4,z) \\ &= \sum_{k=0}^{4} (-)^k \Phi(k,z), \end{aligned}$$

where

$$\Phi(k,z) \equiv \begin{cases} g(z) & \text{if } k = 0, \\ h(z) & \text{if } k = 1, \\ p(z) & \text{if } k = 2, \\ 0 & \text{if } k = 3, \\ q(z) & \text{if } k = 4, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Such unedifying formalism is essentially useless in applications, except as a vehicle for definition. Once you understand why $+$ and $-$ are operators just as $\sum$ and $\int$ are, you can forget the formalism. It doesn't help much.

7.3.3 Linearity

A function $f(z)$ is *linear* iff (if and only if) it has the properties

$$\begin{aligned} f(z_1 + z_2) &= f(z_1) + f(z_2), \\ f(\alpha z) &= \alpha f(z), \\ f(0) &= 0. \end{aligned}$$

The functions $f(z) = 3z$, $f(u,v) = 2u - v$ and $f(z) = 0$ are examples of linear functions. Nonlinear functions include[5] $f(z) = z^2$, $f(u,v) = \sqrt{uv}$, $f(t) = \cos \omega t$, $f(z) = 3z + 1$ and even $f(z) = 1$.

[5]If $3z + 1$ is a *linear expression,* then how is not $f(z) = 3z + 1$ a *linear function?* Answer: it is partly a matter of purposeful definition, partly of semantics. The equation $y = 3x + 1$ plots a line, so the expression $3z + 1$ is literally "linear" in this sense; but the definition has more purpose to it than merely this. When you see the linear expression $3z + 1$, think $3z + 1 = 0$, then $g(z) = 3z = -1$. The $g(z) = 3z$ is linear; the -1 is the constant value it targets. That's the sense of it.

An operator L is linear iff it has the properties

$$\begin{aligned} L(f_1 + f_2) &= Lf_1 + Lf_2, \\ L(\alpha f) &= \alpha L f, \\ L(0) &= 0. \end{aligned}$$

The operators $\sum$, $\int$, $+$, $-$ and ∂ are examples of linear operators. For instance,[6]

$$\frac{d}{dz}[f_1(z) + f_2(z)] = \frac{df_1}{dz} + \frac{df_2}{dz}.$$

Nonlinear operators include multiplication, division and the various trigonometric functions, among others.

7.3.4 Summational and integrodifferential transitivity

Consider the sum

$$S_1 = \sum_{k=a}^{b} \left[\sum_{j=p}^{q} \frac{x^k}{j!} \right].$$

This is a sum of the several values of the expression $x^k/j!$, evaluated at every possible pair (j, k) in the indicated domain. Now consider the sum

$$S_2 = \sum_{j=p}^{q} \left[\sum_{k=a}^{b} \frac{x^k}{j!} \right].$$

This is evidently a sum of the same values, only added in a different order. Apparently $S_1 = S_2$. Reflection along these lines must soon lead the reader to the conclusion that, in general,[7]

$$\sum_k \sum_j f(j, k) = \sum_j \sum_k f(j, k).$$

[6] You don't see d in the list of linear operators? But d in this context is really just another way of writing ∂, so, yes, d is linear, too. See § 4.4.2.

[7] One occasionally encounters a convergent sum like $\sum_{k=1}^{\infty}[(1/k) + (1-k)/k^2]$, whose parts separately fail to converge when reordered $\sum(1/k) + \sum[(1-k)/k^2]$; but in applied mathematics such questions are usually dealt with rationally by considering the facts of the physical model at hand. So long as the thing modeled is clearly grasped, the question of convergence seldom poses a real dilemma in practice. In fact, reordering the sum after the manner of this footnote's example to eliminate the convergence problem is usually the way to go.

Now consider that an integral is just a sum of many elements, and that a derivative is just a difference of two elements. Integrals and derivatives must then have the same transitive property discrete sums have. For example,

$$\begin{aligned}
\int_{v=-\infty}^{\infty}\int_{u=a}^{b} f(u,v)\,du\,dv &= \int_{u=a}^{b}\int_{v=-\infty}^{\infty} f(u,v)\,dv\,du;\\
\int \sum_k f_k(v)\,dv &= \sum_k \int f_k(v)\,dv;\\
\frac{\partial}{\partial v}\int f\,du &= \int \frac{\partial f}{\partial v}\,du.
\end{aligned}$$

In general,

$$L_v L_u f(u,v) = L_u L_v f(u,v), \tag{7.3}$$

where L is any of the linear operators $\sum$, $\int$ or ∂.

7.3.5 Multiple integrals

Consider the function

$$f(u,v) = \frac{u^2}{v}.$$

Such a function would not be plotted as a curved line in a plane, but rather as a curved *surface* in a three-dimensional space. Integrating the function seeks not the area under the curve but rather the volume under the surface:

$$V = \int_{u_1}^{u_2}\int_{v_1}^{v_2} \frac{u^2}{v}\,dv\,du.$$

This is a *double integral.* Inasmuch as it can be written in the form

$$\begin{aligned}
V &= \int_{u_1}^{u_2} g(u)\,du,\\
g(u) &\equiv \int_{v_1}^{v_2} \frac{u^2}{v}\,dv,
\end{aligned}$$

its effect is to cut the area under the surface into flat, upright slices, then the slices crosswise into tall, thin towers. The towers are integrated over v to constitute the slice, then the slices over u to constitute the volume.

In light of § 7.3.4, evidently nothing prevents us from swapping the integrations: u first, then v. Hence

$$V = \int_{v_1}^{v_2}\int_{u_1}^{u_2} \frac{u^2}{v}\,du\,dv.$$

And indeed this makes sense, doesn't it? What difference does it make whether we add the towers by rows first then by columns, or by columns first then by rows? The total volume is the same in any case.

Double integrations arise very frequently in applications. Triple integrations arise about as often. For instance, if $\mu(\mathbf{r}) = \mu(x, y, z)$ represents the variable mass density of some soil,[8] then the total soil mass in some rectangular volume is

$$M = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \mu(x, y, z)\, dz\, dy\, dx.$$

As a concise notational convenience, the last is often written

$$M = \int_V \mu(\mathbf{r})\, d\mathbf{r},$$

where the V stands for "volume" and is understood to imply a triple integration. Similarly for the double integral,

$$V = \int_S f(\boldsymbol{\rho})\, d\boldsymbol{\rho},$$

where the S stands for "surface" and is understood to imply a double integration.

Even more than three nested integrations are possible. If we integrated over time as well as space, the integration would be fourfold. A spatial Fourier transform ([section not yet written]) implies a triple integration; and its inverse, another triple: a sixfold integration altogether. Manifold nesting of integrals is thus not just a theoretical mathematical topic; it arises in sophisticated real-world engineering models. The topic concerns us here for this reason.

7.4 Areas and volumes

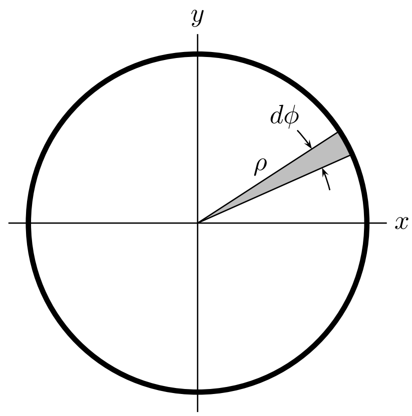
By composing and solving appropriate integrals, one can calculate the perimeters, areas and volumes of interesting common shapes and solids.

7.4.1 The area of a circle

Figure 7.4 depicts an element of a circle's area. The element has wedge

[8] Conventionally the Greek letter ρ not μ is used for density, but it happens that we need the letter ρ for a different purpose later in the paragraph.

Figure 7.4: The area of a circle.



shape, but inasmuch as the wedge is infinitesimally narrow, the wedge is indistinguishable from a triangle of base length $\rho\,d\phi$ and height ρ. The area of such a triangle is $A_{\mathrm{triangle}} = \rho^2\,d\phi/2$. Integrating the many triangles, we find the circle's area to be

$$A_{\mathrm{circle}} = \int_{\phi=-\pi}^{\pi} A_{\mathrm{triangle}} = \int_{-\pi}^{\pi} \frac{\rho^2\,d\phi}{2} = \frac{2\pi\rho^2}{2}. \tag{7.4}$$

(The numerical value of 2π—the circumference or perimeter of the unit circle—we have not calculated yet. We shall calculate it in § 8.9.)

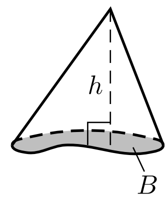
7.4.2 The volume of a cone

One can calculate the volume of any cone (or pyramid) if one knows its base area B and its altitude h measured normal[9] to the base. Refer to Fig. 7.5. A cross-section of a cone, cut parallel to the cone's base, has the same shape the base has but a different scale. If coordinates are chosen such that the altitude h runs in the $\hat{\mathbf{z}}$ direction with $z = 0$ at the cone's vertex, then the cross-sectional area is evidently[10] $(B)(z/h)^2$. For this reason, the cone's

[9] *Normal* means "at right angles."

[10] The fact may admittedly not be evident to the reader at first glance. If it is not yet evident to you, then ponder Fig. 7.5 a moment. Consider what it means to cut parallel to a cone's base a cross-section of the cone, and how cross-sections cut nearer a cone's

Figure 7.5: The volume of a cone.



volume is

$$V_{\text{cone}} = \int_0^h (B)\left(\frac{z}{h}\right)^2 dz = \frac{B}{h^2}\int_0^h z^2\,dz = \frac{B}{h^2}\left(\frac{h^3}{3}\right) = \frac{Bh}{3}. \tag{7.5}$$

7.4.3 The surface area and volume of a sphere

Of a sphere, Fig. 7.6, one wants to calculate both the surface area and the volume. For the surface area, the sphere's surface is sliced vertically down the z axis into narrow constant-ϕ tapered strips (each strip broadest at the sphere's equator, tapering to points at the sphere's $\pm z$ poles) and horizontally across the z axis into narrow constant-θ rings, as in Fig. 7.7. A surface element so produced (seen as shaded in the latter figure) evidently has the area

$$dS = (r\,d\theta)(\rho\,d\phi) = r^2 \sin\theta\,d\theta\,d\phi.$$

---

vertex are smaller though the same shape. What if the base were square? Would the cross-sectional area not be $(B)(z/h)^2$ in that case? What if the base were a right triangle with equal legs—in other words, half a square? What if the base were some other strange shape like the base depicted in Fig. 7.5? Could such a strange shape not also be regarded as a definite, well characterized part of a square? (With a pair of scissors one can cut any shape from a square piece of paper, after all.) Thinking along such lines must soon lead one to the insight that the parallel-cut cross-sectional area of a cone can be nothing other than $(B)(z/h)^2$, regardless of the base's shape.

Figure 7.6: A sphere.

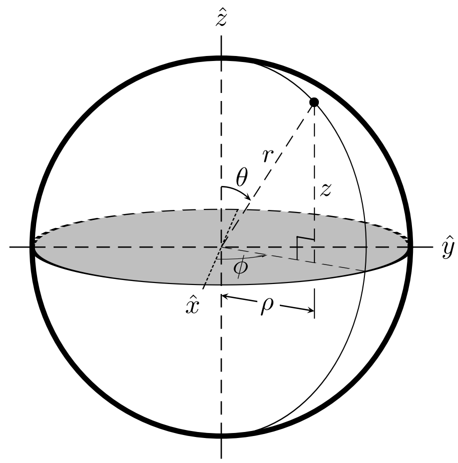
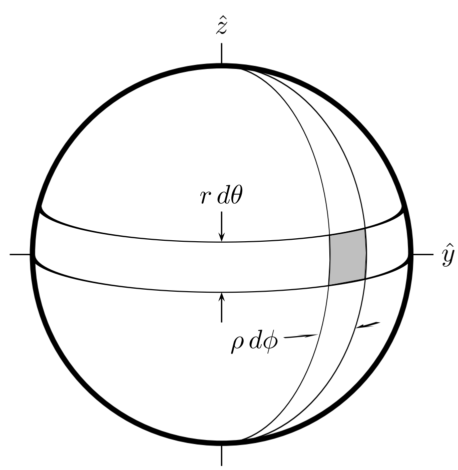


Figure 7.7: An element of the sphere's surface (see Fig. 7.6).



The sphere's total surface area then is the sum of all such elements over the sphere's entire surface:

$$\begin{aligned} S_{\text{sphere}} &= \int_{\phi=-\pi}^{\pi} \int_{\theta=0}^{\pi} dS \\ &= \int_{\phi=-\pi}^{\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi \\ &= r^2 \int_{\phi=-\pi}^{\pi} [-\cos\theta]_0^{\pi} \, d\phi \\ &= r^2 \int_{\phi=-\pi}^{\pi} [2] \, d\phi \\ &= 4\pi r^2, \end{aligned} \tag{7.6}$$

where we have used the fact from Table 7.1 that $\sin\tau = (d/d\tau)(-\cos\tau)$.

Having computed the sphere's surface area, one can find its volume just as § 7.4.1 has found a circle's area—except that instead of dividing the circle into many narrow triangles, one divides the sphere into many narrow *cones,* each cone with base area dS and altitude r, with the vertices of all the cones meeting at the sphere's center. Per (7.5), the volume of one such cone is $V_{\text{cone}} = r\,dS/3$. Hence,

$$V_{\text{sphere}} = \oint_S V_{\text{cone}} = \oint_S \frac{r\,dS}{3} = \frac{r}{3}\oint_S dS = \frac{r}{3} S_{\text{sphere}},$$

where the useful symbol

$$\oint_S$$

indicates *integration over a closed surface.* In light of (7.6), the total volume is

$$V_{\text{sphere}} = \frac{4\pi r^3}{3}. \tag{7.7}$$

(One can compute the same spherical volume more prosaically, without reference to cones, by writing $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$ then integrating $\int_V dV$. The derivation given above, however, is preferred because it lends the additional insight that a sphere can sometimes be viewed as a great cone rolled up about its own vertex. The circular area derivation of § 7.4.1 lends an analogous insight: that a circle can sometimes be viewed as a great triangle rolled up about *its* own vertex.)

7.5 Checking integrations

Dividing 0x46B/0xD = 0x57 with a pencil, how does one check the result?[11] Answer: by multiplying (0x57)(0xD) = 0x46B. Multiplication inverts division. Easier than division, multiplication provides a quick, reliable check.

Likewise, integrating

$$\int_a^b \frac{\tau^2}{2}\,d\tau = \frac{b^3 - a^3}{6}$$

with a pencil, how does one check the result? Answer: by differentiating

$$\left[\frac{\partial}{\partial b}\left(\frac{b^3 - a^3}{6}\right)\right]_{b=\tau} = \frac{\tau^2}{2}.$$

Differentiation inverts integration. Easier than integration, differentiation like multiplication provides a quick, reliable check.

More formally, according to (7.2),

$$S \equiv \int_a^b \frac{df}{d\tau}\,d\tau = f(b) - f(a). \tag{7.8}$$

Differentiating (7.8) with respect to b and a,

$$\begin{aligned} \left.\frac{\partial S}{\partial b}\right|_{b=\tau} &= \frac{df}{d\tau}, \\ \left.\frac{\partial S}{\partial a}\right|_{a=\tau} &= -\frac{df}{d\tau}. \end{aligned} \tag{7.9}$$

Either line of (7.9) can be used to check an integration. Evaluating (7.8) at $b = a$ yields

$$S|_{b=a} = 0, \tag{7.10}$$

which can be used to check further.[12]

As useful as (7.9) and (7.10) are, they nevertheless serve only integrals with variable limits. They are of little use with *definite integrals* like (9.14)

[11] Admittedly, few readers will ever have done much such multidigit *hexadecimal* arithmetic with a pencil, but, hey, go with it. In decimal, it's 1131/13 = 87.

Actually, hexadecimal is just proxy for binary (see Appendix A), and long division in straight binary is kind of fun. If you have never tried it, you might. It is simpler than decimal or hexadecimal division, and it's how computers divide. The insight is worth the trial.

[12] Using (7.10) to check the example, $(b^3 - a^3)/6|_{b=a} = 0$.

below, which lack variable limits to differentiate. However, many or most integrals one meets in practice have or can be given variable limits. Equations (7.9) and (7.10) do serve such *indefinite integrals.*

It is a rare irony of mathematics that, although numerically differentiation is indeed harder than integration, analytically precisely the opposite is true. Analytically, differentiation is the easier. So far the book has introduced only easy integrals, but Ch. 9 will bring much harder ones. Even experienced mathematicians are apt to err in analyzing these. Reversing an integration by taking an easy derivative is thus an excellent way to check a hard-earned integration result.

7.6 Contour integration

To this point we have considered only integrations in which the variable of integration advances in a straight line from one point to another: for instance, $\int_a^b f(\tau)\,d\tau$, in which the function $f(\tau)$ is evaluated at $\tau = a, a + d\tau, a + 2d\tau, \ldots, b$. The integration variable is a real-valued scalar which can do nothing but make a straight line from a to b.

Such is not the case when the integration variable is a vector. Consider the integral

$$S = \int_{\mathbf{r}=\hat{\mathbf{x}}\rho}^{\hat{\mathbf{y}}\rho} (x^2 + y^2)\,d\ell,$$

where $d\ell$ is the infinitesimal length of a step along the path of integration. What does this integral mean? Does it mean to integrate from $\mathbf{r} = \hat{\mathbf{x}}\rho$ to $\mathbf{r} = 0$, then from there to $\mathbf{r} = \hat{\mathbf{y}}\rho$? Or does it mean to integrate along the arc of Fig. 7.8? The two paths of integration begin and end at the same points, but they differ in between, and the integral certainly does not come out the same both ways. Yet many other paths of integration from $\hat{\mathbf{x}}\rho$ to $\hat{\mathbf{y}}\rho$ are possible, not just these two.

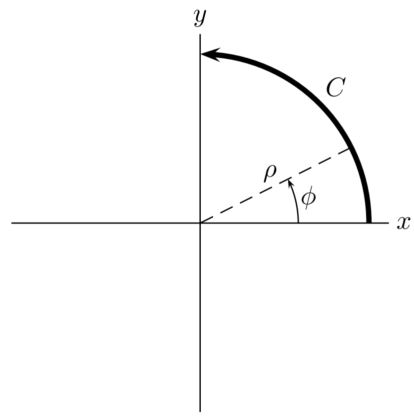
Because multiple paths are possible, we must be more specific:

$$S = \int_C (x^2 + y^2)\,d\ell,$$

where C stands for "contour" and means in this example the specific contour of Fig. 7.8. In the example, $x^2 + y^2 = \rho^2$ (by the Pythagorean theorem) and $d\ell = \rho\,d\phi$, so

$$S = \int_C \rho^2\,d\ell = \int_0^{2\pi/4} \rho^3\,d\phi = \frac{2\pi}{4}\rho^3.$$

Figure 7.8: A contour of integration.



In the example the contour is open, but closed contours which begin and end at the same point are also possible, indeed common. The useful symbol

$$\oint$$

indicates *integration over a closed contour.* It means that the contour ends where it began: the loop is closed. The contour of Fig. 7.8 would be closed, for instance, if it continued to $\mathbf{r} = 0$ and then back to $\mathbf{r} = \hat{\mathbf{x}}\rho$.

Besides applying where the variable of integration is a vector, contour integration applies equally where the variable of integration is a complex scalar. In the latter case some interesting mathematics emerge, as we shall see in §§ 8.6 and 9.5.

7.7 Discontinuities

The polynomials and trigonometrics studied to this point in the book offer flexible means to model many physical phenomena of interest, but one thing they do not model gracefully is the simple discontinuity. Consider a mechanical valve opened at time $t = t_o$. The flow $x(t)$ past the valve is

$$x(t) = \begin{cases} 0, & t < t_o; \\ x_o, & t > t_o. \end{cases}$$

Figure 7.9: The Heaviside unit step $u(t)$.

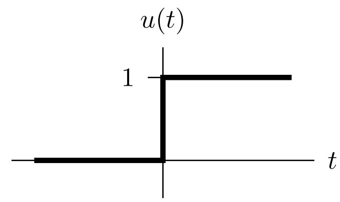
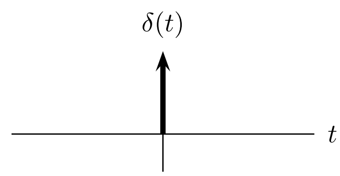


Figure 7.10: The Dirac delta $\delta(t)$.



One can write this more concisely in the form

$$x(t) = u(t - t_o)x_o,$$

where $u(t)$ is the *Heaviside unit step,*

$$u(t) \equiv \begin{cases} 0, & t < 0; \\ 1, & t > 0; \end{cases} \tag{7.11}$$

plotted in Fig. 7.9.

The derivative of the Heaviside unit step is the curious *Dirac delta*

$$\delta(t) \equiv \frac{d}{dt}u(t), \tag{7.12}$$

plotted in Fig. 7.10. This function is zero everywhere except at $t = 0$, where it is infinite, with the property that

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1, \tag{7.13}$$

and the interesting consequence that

$$\int_{-\infty}^{\infty} \delta(t - t_o) f(t)\, dt = f(t_o) \tag{7.14}$$

for any function $f(t)$. (Equation 7.14 is the *sifting property* of the Dirac delta.)[13]

The Dirac delta is defined for vectors, too, such that

$$\int_V \delta(\mathbf{r})\, d\mathbf{r} = 1. \tag{7.15}$$

[13]It seems inadvisable for the narrative to digress at this point to explore $u(z)$ and $\delta(z)$, the unit step and delta of a complex argument, although by means of Fourier analysis ([chapter not yet written]) it could perhaps do so. The book has more pressing topics to treat. For the book's present purpose the interesting action of the two functions is with respect to the real argument t.

In the author's country at least, a sort of debate seems to have run for decades between professional and applied mathematicians over the Dirac delta $\delta(t)$. Some professional mathematicians seem to have objected that $\delta(t)$ is not a function, inasmuch as it lacks certain properties common to functions as they define them [19, § 2.4][9]. From the applied point of view the objection is admittedly a little hard to understand, until one realizes that it is more a dispute over methods and definitions than over facts. What the professionals seem to be saying is that $\delta(t)$ does not fit as neatly as they would like into the abstract mathematical framework they had established for functions in general before Paul Dirac came along in 1930 [3, "Paul Dirac," 05:48, 25 May 2006] and slapped his disruptive $\delta(t)$ down on the table. The objection is not so much that $\delta(t)$ is not allowed as it is that professional mathematics for years after 1930 lacked a fully coherent theory for it.

It's a little like the six-fingered man in Goldman's *The Princess Bride* [13]. If I had established a definition of "nobleman" which subsumed "human," whose relevant traits in my definition included five fingers on each hand, when the six-fingered Count Rugen appeared on the scene, then you would expect me to adapt my definition, wouldn't you? By my preëxisting definition, strictly speaking, the six-fingered count is "not a nobleman;" but such exclusion really tells one more about flaws in the definition than it does about the count.

Whether the professional mathematician's definition of the *function* is flawed, of course, is not for this writer to judge. Even if not, however, the fact of the Dirac delta dispute, coupled with the difficulty we applied mathematicians experience in trying to understand the reason the dispute even exists, has unfortunately surrounded the Dirac delta with a kind of mysterious aura, an elusive sense that $\delta(t)$ hides subtle mysteries—when what it really hides is an internal discussion of words and means among the professionals. The professionals who had established the theoretical framework before 1930 justifiably felt reluctant to throw the whole framework away because some scientists and engineers like us came along one day with a useful new function which didn't quite fit, but that was the professionals' problem not ours. To us the Dirac delta $\delta(t)$ is just a function. The internal discussion of words and means, we leave to the professionals, who know whereof they speak.

7.8 Remarks (and exercises)

The concept of the integral is relatively simple once grasped, but its implications are broad, deep and hard. This chapter is short. One reason introductory calculus texts are so long is that they include many, many pages of integral examples and exercises. The reader who desires a gentler introduction to the integral might consult among others the textbook recommended in the chapter's introduction.

Even if this book is not an instructional textbook, it seems not meet that it should include no exercises at all here. Here are a few. Some of them do need material from later chapters, so you should not expect to be able to complete them all now. The harder ones are marked with *asterisks. Work the exercises if you like.

1. Evaluate (a) $\int_0^x \tau\,d\tau$; (b) $\int_0^x \tau^2\,d\tau$. (Answer: $x^2/2$; $x^3/3$.)

2. Evaluate (a) $\int_1^x (1/\tau^2)\,d\tau$; (b) $\int_a^x 3\tau^{-2}\,d\tau$; (c) $\int_a^x C\tau^n\,d\tau$; (d) $\int_0^x (a_2\tau^2 + a_1\tau)\,d\tau$; *(e) $\int_1^x (1/\tau)\,d\tau$.

3. *Evaluate (a) $\int_0^x \sum_{k=0}^{\infty} \tau^k\,d\tau$; (b) $\sum_{k=0}^{\infty} \int_0^x \tau^k\,d\tau$; (c) $\int_0^x \sum_{k=0}^{\infty} (\tau^k/k!)\,d\tau$.

4. Evaluate $\int_0^x \exp \alpha\tau\,d\tau$.

5. Evaluate (a) $\int_{-2}^{5} (3\tau^2 - 2\tau^3)\,d\tau$; (b) $\int_5^{-2} (3\tau^2 - 2\tau^3)\,d\tau$. Work the exercise by hand in hexadecimal and give the answer in hexadecimal.

6. Evaluate $\int_1^{\infty} (3/\tau^2)\,d\tau$.

7. *Evaluate the integral of the example of § 7.6 along the alternate contour suggested there, from $\hat{\mathbf{x}}\rho$ to 0 to $\hat{\mathbf{y}}\rho$.

8. Evaluate (a) $\int_0^x \cos\omega\tau\,d\tau$; (b) $\int_0^x \sin\omega\tau\,d\tau$; *(c) [22, § 8-2] $\int_0^x \tau\sin\omega\tau\,d\tau$.

9. *Evaluate [22, § 5-6] (a) $\int_1^x \sqrt{1+2\tau}\,d\tau$; (b) $\int_x^a [(\cos\sqrt{\tau})/\sqrt{\tau}]\,d\tau$.

10. *Evaluate [22, back endpaper] (a) $\int_0^x [1/(1+\tau^2)]\,d\tau$ (answer: $\arctan x$); (b) $\int_0^x [(4+i3)/\sqrt{2-3\tau^2}]\,d\tau$ (hint: the answer involves another inverse trigonometric).

11. **Evaluate (a) $\int_{-\infty}^x \exp[-\tau^2/2]\,d\tau$; (b) $\int_{-\infty}^{\infty} \exp[-\tau^2/2]\,d\tau$.

The last exercise in particular requires some experience to answer. Moreover, it requires a developed sense of applied mathematical style to put the answer in a pleasing form (the right form for part b is very different from that for part a). Some of the easier exercises, of course, you should be able to answer right now.

The point of the exercises is to illustrate how hard integrals can be to solve, and in fact how easy it is to come up with an integral which no one really knows how to solve very well. Some solutions to the same integral are better than others (easier to manipulate, faster to numerically calculate, etc.) yet not even the masters can solve them all in practical ways. On the other hand, integrals which arise in practice often can be solved very well with sufficient cleverness—and the more cleverness you develop, the more such integrals you can solve. The ways to solve them are myriad. The mathematical art of solving diverse integrals is well worth cultivating.

Chapter 9 introduces some of the basic, most broadly useful integral-solving techniques. Before addressing techniques of integration, however, as promised earlier we turn our attention in Chapter 8 back to the derivative, applied in the form of the Taylor series.

Chapter 8

The Taylor series

The Taylor series is a power series which fits a function in a limited domain neighborhood. Fitting a function in such a way brings two advantages:

- it lets us take derivatives and integrals in the same straightforward way (4.20) we take them with any power series; and
- it implies a simple procedure to calculate the function numerically.

This chapter introduces the Taylor series and some of its incidents. It also derives Cauchy's integral formula. The chapter's early sections prepare the ground for the treatment of the Taylor series proper in § 8.3.[1]

8.1 The power series expansion of $1/(1-z)^{n+1}$

Before approaching the Taylor series proper in § 8.3, we shall find it both interesting and useful to demonstrate that

$$\frac{1}{(1-z)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{n} z^k \tag{8.1}$$

for $n \geq 0$, $|z| < 1$. The demonstration comes in three stages. Of the three, it is the second stage (§ 8.1.2) which actually proves (8.1). The first stage

[1]Because even at the applied level the proper derivation of the Taylor series involves mathematical induction, analytic continuation and the matter of convergence domains, no balance of rigor the chapter might strike seems wholly satisfactory. The chapter errs maybe toward too much rigor; for, with a little less, most of §§ 8.1, 8.2 and 8.4 would cease to be necessary. The impatient reader should at least examine (8.1) from § 8.1, and should consider reading about analytic continuation in § 8.4, but otherwise to skip the three sections might not be an unreasonable way to shorten the chapter.

(§ 8.1.1) comes up with the formula for the second stage to prove. The third stage (§ 8.1.3) establishes the sum's convergence.

8.1.1 The formula

In § 2.6.3 we found that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots$$

for $|z| < 1$. What about $1/(1-z)^2$, $1/(1-z)^3$, $1/(1-z)^4$, and so on? By the long-division procedure of Table 2.4, one can calculate the first few terms of $1/(1-z)^2$ to be

$$\frac{1}{(1-z)^2} = \frac{1}{1-2z+z^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

whose coefficients $1, 2, 3, 4, \ldots$ happen to be the numbers down the first diagonal of Pascal's triangle (Fig. 4.2 on page 71; see also Fig. 4.1). Dividing $1/(1-z)^3$ seems to produce the coefficients $1, 3, 6, \mathrm{0xA}, \ldots$ down the second diagonal; dividing $1/(1-z)^4$, the coefficients down the third. A curious pattern seems to emerge, worth investigating more closely. The pattern recommends the conjecture (8.1).

To motivate the conjecture a bit more formally (though without actually proving it yet), suppose that $1/(1-z)^{n+1}$, $n \geq 0$, is expandable in the power series

$$\frac{1}{(1-z)^{n+1}} = \sum_{k=0}^{\infty} a_{nk} z^k, \tag{8.2}$$

where the a_{nk} are coefficients to be determined. Multiplying by $1 - z$, we have that

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} (a_{nk} - a_{n(k-1)}) z^k.$$

This is to say that

$$a_{(n-1)k} = a_{nk} - a_{n(k-1)},$$

or in other words that

$$a_{n(k-1)} + a_{(n-1)k} = a_{nk}. \tag{8.3}$$

Thinking of Pascal's triangle, (8.3) reminds one of (4.5), transcribed here in the symbols

$$\binom{m-1}{j-1} + \binom{m-1}{j} = \binom{m}{j}, \tag{8.4}$$

except that (8.3) is not $a_{(m-1)(j-1)} + a_{(m-1)j} = a_{mj}$.

Various changes of variable are possible to make (8.4) better match (8.3). We might try at first a few false ones, but eventually the change

$$\begin{aligned} n+k &\leftarrow m, \\ k &\leftarrow j, \end{aligned}$$

recommends itself. Thus changing in (8.4) gives

$$\binom{n+k-1}{k-1} + \binom{n+k-1}{k} = \binom{n+k}{k}.$$

Transforming according to the rule (4.3), this is

$$\binom{n+(k-1)}{n} + \binom{(n-1)+k}{n-1} = \binom{n+k}{n}, \tag{8.5}$$

which fits (8.3) perfectly. Hence we conjecture that

$$a_{nk} = \binom{n+k}{n}, \tag{8.6}$$

which coefficients, applied to (8.2), yield (8.1).

Equation (8.1) is thus suggestive. It works at least for the important case of $n = 0$; this much is easy to test. In light of (8.3), it seems to imply a relationship between the $1/(1-z)^{n+1}$ series and the $1/(1-z)^n$ series for any n. But *to seem* is not *to be.* At this point, all we can say is that (8.1) seems right. We shall establish that it is right in the next subsection.

8.1.2 The proof by induction

Equation (8.1) is proven by induction as follows. Consider the sum

$$S_n \equiv \sum_{k=0}^{\infty} \binom{n+k}{n} z^k. \tag{8.7}$$

Multiplying by $1-z$ yields

$$(1-z)S_n = \sum_{k=0}^{\infty} \left[\binom{n+k}{n} - \binom{n+(k-1)}{n} \right] z^k.$$

Per (8.5), this is

$$(1-z)S_n = \sum_{k=0}^{\infty} \binom{(n-1)+k}{n-1} z^k. \tag{8.8}$$

Now suppose that (8.1) is true for $n = i - 1$ (where i denotes an integer rather than the imaginary unit):

$$\frac{1}{(1-z)^i} = \sum_{k=0}^{\infty} \binom{(i-1)+k}{i-1} z^k. \tag{8.9}$$

In light of (8.8), this means that

$$\frac{1}{(1-z)^i} = (1-z)S_i.$$

Dividing by $1 - z$,

$$\frac{1}{(1-z)^{i+1}} = S_i.$$

Applying (8.7),

$$\frac{1}{(1-z)^{i+1}} = \sum_{k=0}^{\infty} \binom{i+k}{i} z^k. \tag{8.10}$$

Evidently (8.9) implies (8.10). In other words, if (8.1) is true for $n = i - 1$, then it is also true for $n = i$. Thus *by induction,* if it is true for any one n, then it is also true for all greater n.

The "if" in the last sentence is important. Like all inductions, this one needs at least one *start case* to be valid (many inductions actually need a consecutive pair of start cases). The $n = 0$ supplies the start case

$$\frac{1}{(1-z)^{0+1}} = \sum_{k=0}^{\infty} \binom{k}{0} z^k = \sum_{k=0}^{\infty} z^k,$$

which per (2.29) we know to be true.

8.1.3 Convergence

The question remains as to the domain over which the sum (8.1) converges.[2] To answer the question, consider that per (4.9),

$$\binom{m}{j} = \frac{m}{m-j} \binom{m-1}{j}$$

[2]The meaning of the verb *to converge* may seem clear enough from the context and from earlier references, but if explanation here helps: a series converges if and only if it approaches a specific, finite value after many terms. A more rigorous way of saying the same thing is as follows: the series

$$S = \sum_{k=0}^{\infty} \tau_k$$

for any integers $m > 0$ and j. With the substitution $n + k \leftarrow m$, $n \leftarrow j$, this means that

$$\binom{n+k}{n} = \frac{n+k}{k}\binom{n+(k-1)}{n},$$

or more tersely,

$$a_{nk} = \frac{n+k}{k} a_{n(k-1)},$$

where

$$a_{nk} \equiv \binom{n+k}{n}$$

are the coefficients of the power series (8.1). Rearranging factors,

$$\frac{a_{nk}}{a_{n(k-1)}} = \frac{n+k}{k} = 1 + \frac{n}{k}. \tag{8.11}$$

Multiplying (8.11) by z^k/z^{k-1} gives the ratio

$$\frac{a_{nk}z^k}{a_{n(k-1)}z^{k-1}} = \left(1 + \frac{n}{k}\right) z,$$

which is to say that the kth term of (8.1) is $(1 + n/k)z$ times the $(k-1)$th term. So long as the criterion[3]

$$\left|\left(1 + \frac{n}{k}\right) z\right| \le 1 - \delta$$

is satisfied for all sufficiently large $k > K$—where $0 < \delta \ll 1$ is a small positive constant—then the series evidently converges (see § 2.6.3 and eqn. 3.20).

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converges iff (if and only if), for all possible positive constants ϵ, there exists a $K \ge 0$ such that

$$\left|\sum_{k=K}^{n} \tau_k\right| < \epsilon,$$

for all $n \ge K - 1$. (Of course it is also required that the τ_k be finite, but you knew that already.)

[3]Although one need not ask the question to understand the proof, the reader may nevertheless wonder why the simpler $|(1 + n/k)z| < 1$ is not given as a criterion. The surprising answer is that not all series $\sum \tau_k$ with $|\tau_k/\tau_{k-1}| < 1$ converge! For example, the extremely simple $\sum 1/k$ does not converge. As we see however, all series $\sum \tau_k$ with $|\tau_k/\tau_{k-1}| < 1 - \delta$ do converge. The distinction is subtle but rather important.

The really curious reader may now ask why $\sum 1/k$ does not converge. Answer: it *majorizes* $\int_1^x (1/\tau)\, d\tau = \ln x$. See (5.7) and § 8.8.

But we can bind $1+n/k$ as close to unity as desired by making K sufficiently large, so to meet the criterion it suffices that

$$|z| < 1. \tag{8.12}$$

The bound (8.12) thus establishes a sure convergence domain for (8.1).

8.1.4 General remarks on mathematical induction

We have proven (8.1) by means of a mathematical induction. The virtue of induction as practiced in § 8.1.2 is that it makes a logically clean, airtight case for a formula. Its vice is that it conceals the subjective process which has led the mathematician to consider the formula in the first place. Once you obtain a formula somehow, maybe you can prove it by induction; but the induction probably does not help you to obtain the formula! A good inductive proof usually begins by motivating the formula proven, as in § 8.1.1.

Richard W. Hamming once said of mathematical induction,

> The theoretical difficulty the student has with mathematical induction arises from the reluctance to ask seriously, "How could I prove a formula for an infinite number of cases when I know that testing a finite number of cases is not enough?" Once you really face this question, you will understand the ideas behind mathematical induction. It is only when you grasp the problem clearly that the method becomes clear. [14, § 2.3]

Hamming also wrote,

> The function of rigor is mainly critical and is seldom constructive. Rigor is the hygiene of mathematics, which is needed to protect us against careless thinking. [14, § 1.6]

The applied mathematician may tend to avoid rigor for which he finds no immediate use, but he does not disdain mathematical rigor on principle. The style lies in exercising rigor at the right level for the problem at hand. Hamming, a professional mathematician who sympathized with the applied mathematician's needs, wrote further,

> Ideally, when teaching a topic the degree of rigor should follow the student's perceived need for it... It is necessary to require a gradually rising level of rigor so that when faced with a real

> need for it you are not left helpless. As a result, [one cannot teach] a uniform level of rigor, but rather a gradually rising level. Logically, this is indefensible, but psychologically there is little else that can be done. [14, § 1.6]

Applied mathematics holds that the practice *is* defensible, on the ground that the math serves the model; but Hamming nevertheless makes a pertinent point.

Mathematical induction is a broadly applicable technique for constructing mathematical proofs. We shall not always write inductions out as explicitly in this book as we have done in the present section—often we shall leave the induction as an implicit exercise for the interested reader—but this section's example at least lays out the general pattern of the technique.

8.2 Shifting a power series' expansion point

One more question we should treat before approaching the Taylor series proper in § 8.3 concerns the shifting of a power series' expansion point. How can the expansion point of the power series

$$f(z) = \sum_{k=K}^{\infty} (a_k)(z - z_o)^k, \quad K \le 0, \tag{8.13}$$

be shifted from $z = z_o$ to $z = z_1$?

The first step in answering the question is straightforward: one rewrites (8.13) in the form

$$f(z) = \sum_{k=K}^{\infty} (a_k)([z - z_1] - [z_o - z_1])^k,$$

then changes the variables

$$\begin{aligned} w &\leftarrow \frac{z - z_1}{z_o - z_1}, \\ c_k &\leftarrow [-(z_o - z_1)]^k a_k, \end{aligned} \tag{8.14}$$

to obtain

$$f(z) = \sum_{k=K}^{\infty} (c_k)(1 - w)^k. \tag{8.15}$$

Separating the $k < 0$ terms from the $k \geq 0$ terms in (8.15), we have that

$$\begin{aligned} f(z) &= f_-(z) + f_+(z), \\ f_-(z) &\equiv \sum_{k=0}^{-(K+1)} \frac{c_{[-(k+1)]}}{(1-w)^{k+1}}, \\ f_+(z) &\equiv \sum_{k=0}^{\infty} (c_k)(1-w)^k. \end{aligned} \tag{8.16}$$

Of the two subseries, the $f_-(z)$ is expanded term by term using (8.1), after which combining like powers of w yields the form

$$\begin{aligned} f_-(z) &= \sum_{k=0}^{\infty} q_k w^k, \\ q_k &\equiv \sum_{n=0}^{-(K+1)} (c_{[-(n+1)]}) \begin{pmatrix} n+k \\ n \end{pmatrix}. \end{aligned} \tag{8.17}$$

The $f_+(z)$ is even simpler to expand: one need only multiply the series out term by term per (4.12), combining like powers of w to reach the form

$$\begin{aligned} f_+(z) &= \sum_{k=0}^{\infty} p_k w^k, \\ p_k &\equiv \sum_{n=k}^{\infty} (c_n) \begin{pmatrix} n \\ k \end{pmatrix}. \end{aligned} \tag{8.18}$$

Equations (8.13) through (8.18) show how to shift a power series' expansion point; that is, how to calculate the coefficients of a power series for $f(z)$ about $z = z_1$, given those of a power series about $z = z_o$. Notice that—unlike the original, $z = z_o$ power series—the new, $z = z_1$ power series has terms $(z - z_1)^k$ only for $k \geq 0$. At the price per (8.12) of restricting the convergence domain to $|w| < 1$, shifting the expansion point away from the pole at $z = z_o$ has resolved the $k < 0$ terms.

The method fails if $z = z_1$ happens to be a pole or other nonanalytic point of $f(z)$. The convergence domain vanishes as z_1 approaches such a forbidden point. (Examples of such forbidden points include $z = 0$ in $h(z) = 1/z$ and in $g(z) = \sqrt{z}$. See §§ 8.4 through 8.6.)

The attentive reader might observe that we have formally established the convergence neither of $f_-(z)$ in (8.17) nor of $f_+(z)$ in (8.18). Regarding the former convergence, that of $f_-(z)$, we have strategically framed the problem so that one needn't worry about it, running the sum in (8.13) from the finite $k = K \leq 0$ rather than from the infinite $k = -\infty$; and since according to (8.12) each term of the original $f_-(z)$ of (8.16) converges for $|w| < 1$, the reconstituted $f_-(z)$ of (8.17) safely converges in the same domain. The latter convergence, that of $f_+(z)$, is harder to establish in the abstract because that subseries has an infinite number of terms. As we shall see by pursuing a different line of argument in § 8.3, however, the $f_+(z)$ of (8.18) can be nothing other than the Taylor series about $z = z_1$ of the function $f_+(z)$ in any case, enjoying the same convergence domain any such Taylor series enjoys.

The applied mathematician will normally as a matter of course establish convergence domains for the specific series he calculates. However, regarding the abstract, unspecified series $f(z)$, although one could perhaps formally establish convergence under suitably contrived conditions by a laborious variation on the argument of § 8.1.3, or perhaps by other means,[4] nevertheless for a book of applied mathematics such lines of investigation seem unwise to pursue (pure mathematics has to be better for something, after all). We shall speak no more of the abstract convergence matter here.

8.3 Expanding functions in Taylor series

Having prepared the ground, we now stand in a position to treat the Taylor series proper. The treatment begins with a question: if you had to express some function $f(z)$ by a power series

$$f(z) = \sum_{k=0}^{\infty} (a_k)(z - z_o)^k,$$

how would you do it? The procedure of § 8.1 worked well enough in the case of $f(z) = 1/(1-z)^{n+1}$, but it is not immediately obvious that the same procedure works more generally. What if $f(z) = \sin z$, for example?[5]

Fortunately a different way to attack the power-series expansion problem is known. It works by asking the question: what power series most resembles $f(z)$ in the immediate neighborhood of $z = z_o$? To resemble $f(z)$, the desired power series should have $a_0 = f(z_o)$; otherwise it would not have the right value at $z = z_o$. Then it should have $a_1 = f'(z_o)$ for the right

[4] If any reader happens to know a *concise* argument which establishes convergence usefully in the abstract case, the author would be grateful to receive it.

[5] The actual Taylor series for $\sin z$ is given in § 8.7.

slope. Then, $a_2 = f''(z_o)/2$ for the right second derivative, and so on. With this procedure,

$$f(z) = \sum_{k=0}^{\infty} \left(\left. \frac{d^k f}{dz^k} \right|_{z=z_o} \right) \frac{(z-z_o)^k}{k!}. \tag{8.19}$$

Equation (8.19) is the *Taylor series.* Where it converges, it has all the same derivatives $f(z)$ has, so if $f(z)$ is infinitely differentiable then the Taylor series is an exact representation of the function.[6]

The Taylor series is not guaranteed to converge outside some neighborhood near $z = z_o$.

8.4 Analytic continuation

As earlier mentioned in § 2.12.3, an *analytic function* is a function which is infinitely differentiable in the domain neighborhood of interest—or maybe more appropriately for our applied purpose—a function expressible as a Taylor series in that neighborhood. As we have seen, only one Taylor series

[6]Further proof details may be too tiresome to inflict on applied mathematical readers. However, for readers who want a little more rigor nevertheless, the argument goes briefly as follows. Consider an infinitely differentiable function $F(z)$ and its Taylor series $f(z)$ about z_o. Let $\Delta F(z) \equiv F(z) - f(z)$ be the part of $F(z)$ not representable as a Taylor series about z_o.

If $\Delta F(z)$ is the part of $F(z)$ not representable as a Taylor series, then $\Delta F(z_o)$ and all its derivatives at z_o must be identically zero (otherwise by the Taylor series formula of eqn. 8.19, one could construct a nonzero Taylor series for $\Delta F(z_o)$ from the nonzero derivatives). However, if $F(z)$ is infinitely differentiable and if *all* the derivatives of $\Delta F(z)$ are zero at $z = z_o$, then by the unbalanced definition of the derivative from § 4.4, all the derivatives must also be zero at $z = z_o \pm \epsilon$, hence also at $z = z_o \pm 2\epsilon$, and so on. This means that $\Delta F(z) = 0$. In other words, there is no part of $F(z)$ not representable as a Taylor series.

The interested reader can fill the details in, but basically that is how the more rigorous proof goes. The reason the rigorous proof is confined to a footnote is not a deprecation of rigor as such. It is a deprecation of rigor which serves little purpose in applications. Applied mathematicians normally regard mathematical functions to be imprecise analogs of physical quantities of interest. Since the functions are imprecise analogs in any case, the applied mathematician is logically free implicitly to *define* the functions he uses as Taylor series in the first place; that is, to restrict the set of infinitely differentiable functions used in the model to the subset of such functions representable as Taylor series. With such an implicit definition, whether there actually exist any infinitely differentiable functions not representable as Taylor series is more or less beside the point.

In applied mathematics, the definitions serve the model, not the other way around.

(It is entertaining to consider [28, "Extremum"] the Taylor series of the function $\sin[1/x]$—although in practice this particular function is readily expanded after the obvious change of variable $u \leftarrow 1/x$.)

about z_o is possible for a given function $f(z)$:

$$f(z) = \sum_{k=0}^{\infty} (a_k)(z - z_o)^k.$$

However, nothing prevents one from transposing the series to a different expansion point by the method of § 8.2, except that the transposed series may enjoy a different convergence domain. Evidently so long as the original, $z = z_o$ series and the transposed, $z = z_1$ series have convergence domains which overlap, they describe the same underlying analytic function.

Since an analytic function $f(z)$ is infinitely differentiable and enjoys a unique Taylor expansion $f_o(z - z_o)$ about each point z_o in its domain, it follows that if two Taylor series $f_1(z - z_1)$ and $f_2(z - z_2)$ find even a small neighborhood $|z - z_o| < \epsilon$ which lies in the domain of both, then the two can both be transposed to the common $z = z_o$ expansion point. If the two are found to have the same Taylor series there, then f_1 and f_2 both represent the same function. Moreover, if a series f_3 is found whose domain overlaps that of f_2, then a series f_4 whose domain overlaps that of f_3, and so on, and if each pair in the chain matches at least in a small neighborhood in its region of overlap, then the whole chain of overlapping series necessarily represent the same underlying analytic function $f(z)$. The series f_1 and the series f_n represent the same analytic function even if their domains do not directly overlap at all.

This is a manifestation of the principle of *analytic continuation.* The principle holds that if two analytic functions are the same within some domain neighborhood $|z - z_o| < \epsilon$, then they are the same everywhere.[7] Observe however that the principle fails at poles and other nonanalytic points, because the function is not differentiable there.

The result of § 8.2, which shows general power series to be expressible as Taylor series except at their poles and other nonanalytic points, extends the analytic continuation principle to cover power series in general.

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[7]The writer hesitates to mention that he is given to understand [24] that the domain neighborhood can technically be reduced to a domain contour of nonzero length but zero width. Having never met a significant application of this extension of the principle, the writer has neither researched the extension's proof nor asserted its truth. He does not especially recommend that the reader worry over the point. The domain neighborhood $|z - z_o| < \epsilon$ suffices.

8.5 Branch points

The function $g(z) = \sqrt{z}$ is an interesting, troublesome function. Its derivative is $dg/dz = 1/2\sqrt{z}$, so even though the function is finite at $z = 0$, its derivative is not finite there. Evidently $g(z)$ has a nonanalytic point at $z = 0$, yet the point is not a pole. What is it?

We call it a *branch point.* The defining characteristic of the branch point is that, given a function $f(z)$ with a branch point at $z = z_o$, if one encircles once alone the branch point by a closed contour in the Argand domain plane, while simultaneously tracking $f(z)$ in the Argand range plane—and if one demands that z and $f(z)$ move smoothly, that neither suddenly skip from one spot to another—then one finds that $f(z)$ ends in a different place than it began, even though z itself has returned precisely to its own starting point. The range contour remains open even though the domain contour is closed.

> In complex analysis, a branch point may be thought of informally as a point z_o at which a "multiple-valued function" changes values when one winds once around z_o. [3, "Branch point," 18:10, 16 May 2006]

An analytic function like $g(z) = \sqrt{z}$ having a branch point evidently is not single-valued. It is multiple-valued. For a single z more than one distinct $g(z)$ is possible.

An analytic function like $h(z) = 1/z$, by contrast, is single-valued even though it has a pole. This function does not suffer the syndrome described. When a domain contour encircles a pole, the corresponding range contour is properly closed. Poles do not cause their functions to be multiple-valued and thus are not branch points.

Evidently $f(z) \equiv (z - z_o)^a$ has a branch point at $z = z_o$ if and only if a is not an integer. If $f(z)$ does have a branch point—if a is not an integer—then the mathematician must draw a distinction between $z_1 = \rho e^{i\phi}$ and $z_2 = \rho e^{i(\phi+2\pi)}$, *even though the two are exactly the same number.* Indeed $z_1 = z_2$, but paradoxically $f(z_1) \neq f(z_2)$.

This is difficult. It is confusing, too, until one realizes that the fact of a branch point says nothing whatsoever about the argument z. As far as z is concerned, there really is no distinction between $z_1 = \rho e^{i\phi}$ and $z_2 = \rho e^{i(\phi+2\pi)}$—none at all. What draws the distinction is the multiple-valued function $f(z)$ which uses the argument.

It is as though I had a mad colleague who called me Thaddeus Black, until one day I happened to walk past behind his desk (rather than in front as I usually did), whereupon for some reason he began calling me Gorbag

Pfufnik. I had not changed at all, but now the colleague calls me by a different name. The change isn't really in me, is it? It's in my colleague, who seems to suffer a branch point. If it is important to me to be sure that my colleague really is addressing me when he cries, "Pfufnik!" then I had better keep a running count of how many times I have turned about his desk, hadn't I, even though the number of turns is personally of no import to me.

The usual analysis strategy when one encounters a branch point is simply to avoid the point. Where an integral follows a closed contour as in § 8.6, the strategy is to compose the contour to exclude the branch point, to shut it out. Such a strategy of avoidance usually prospers.[8]

8.6 Cauchy's integral formula

In § 7.6 we considered the problem of vector contour integration, in which the value of an integration depends not only on the integration's endpoints but also on the path, or *contour,* over which the integration is done, as in Fig. 7.8. Because real scalars are confined to a single line, no alternate choice of path is possible where the variable of integration is a real scalar, so the contour problem does not arise in that case. It does however arise where the variable of integration is a *complex* scalar, because there again different paths are possible. Refer to the Argand plane of Fig. 2.5.

Consider the integral

$$S_n = \int_{z_1}^{z_2} z^{n-1}\,dz. \tag{8.20}$$

If z were always a real number, then by the antiderivative (§ 7.2) this integral would evaluate to $(z_2^n - z_1^n)/n$; or, in the case of $n = 0$, to $\ln(z_2/z_1)$. Inasmuch as z is complex, however, the correct evaluation is less obvious. To evaluate the integral sensibly in the latter case, one must consider some specific path of integration in the Argand plane. One must also consider the meaning of the symbol dz.

[8]Traditionally associated with branch points in complex variable theory are the notions of *branch cuts* and *Riemann sheets.* These ideas are interesting, but are not central to the analysis as developed in this book and are not covered here. The interested reader might consult a book on complex variables or advanced calculus like [15], among many others.

8.6.1 The meaning of the symbol dz

The symbol dz represents an infinitesimal step in some direction in the Argand plane:

$$\begin{aligned} dz &= [z + dz] - [z] \\ &= \left[(\rho + d\rho)e^{i(\phi + d\phi)}\right] - \left[\rho e^{i\phi}\right] \\ &= \left[(\rho + d\rho)e^{i\,d\phi}e^{i\phi}\right] - \left[\rho e^{i\phi}\right] \\ &= \left[(\rho + d\rho)(1 + i\,d\phi)e^{i\phi}\right] - \left[\rho e^{i\phi}\right]. \end{aligned}$$

Since the product of two infinitesimals is negligible even on infinitesimal scale, we can drop the $d\rho\,d\phi$ term.[9] After canceling finite terms, we are left with the peculiar but excellent formula

$$dz = (d\rho + i\rho\,d\phi)e^{i\phi}. \tag{8.21}$$

8.6.2 Integrating along the contour

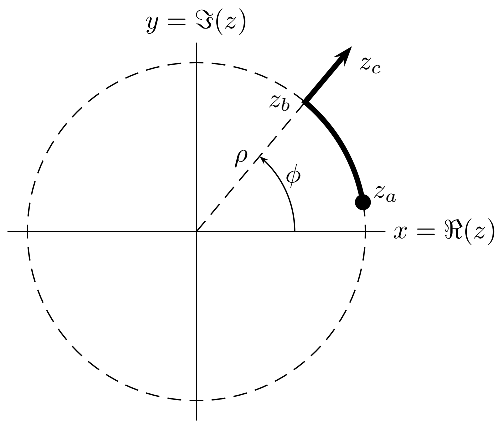
Now consider the integration (8.20) along the contour of Fig. 8.1. Integrat-

[9]The dropping of second-order infinitesimals like $d\rho\,d\phi$, added to first order infinitesimals like $d\rho$, is a standard calculus technique. One cannot *always* drop them, however. Occasionally one encounters a sum in which not only do the finite terms cancel, but also the first-order infinitesimals. In such a case, the second-order infinitesimals dominate and cannot be dropped. An example of the type is

$$\lim_{\epsilon \to 0} \frac{(1-\epsilon)^3 + 3(1+\epsilon) - 4}{\epsilon^2} = \lim_{\epsilon \to 0} \frac{(1 - 3\epsilon + 3\epsilon^2) + (3 + 3\epsilon) - 4}{\epsilon^2} = 3.$$

One typically notices that such a case has arisen when the dropping of second-order infinitesimals has left an ambiguous 0/0. To fix the problem, you simply go back to the step where you dropped the infinitesimal and you restore it, then you proceed from there. Otherwise there isn't much point in carrying second-order infinitesimals around. In the relatively uncommon event that you need them, you'll know it. The math itself will tell you.

Figure 8.1: A contour of integration in the Argand plane, in two parts: constant-ρ (z_a to z_b); and constant-ϕ (z_b to z_c).



ing along the constant-ϕ segment,

$$
\begin{aligned}
\int_{z_b}^{z_c} z^{n-1}\,dz &= \int_{\rho_b}^{\rho_c} (\rho e^{i\phi})^{n-1}(d\rho + i\rho\,d\phi)e^{i\phi} \\
&= \int_{\rho_b}^{\rho_c} (\rho e^{i\phi})^{n-1}(d\rho)e^{i\phi} \\
&= e^{in\phi}\int_{\rho_b}^{\rho_c} \rho^{n-1}d\rho \\
&= \frac{e^{in\phi}}{n}\,(\rho_c^n - \rho_b^n) \\
&= \frac{z_c^n - z_b^n}{n}.
\end{aligned}
$$

Integrating along the constant-ρ arc,

$$\begin{aligned}
\int_{z_a}^{z_b} z^{n-1}\,dz &= \int_{\phi_a}^{\phi_b} (\rho e^{i\phi})^{n-1}(d\rho + i\rho\,d\phi)e^{i\phi} \\
&= \int_{\phi_a}^{\phi_b} (\rho e^{i\phi})^{n-1}(i\rho\,d\phi)e^{i\phi} \\
&= i\rho^n \int_{\phi_a}^{\phi_b} e^{in\phi}\,d\phi \\
&= \frac{i\rho^n}{in}\left(e^{in\phi_b} - e^{in\phi_a}\right) \\
&= \frac{z_b^n - z_a^n}{n}.
\end{aligned}$$

Adding the two parts of the contour integral, we have that

$$\int_{z_a}^{z_c} z^{n-1}\,dz = \frac{z_c^n - z_a^n}{n},$$

surprisingly the same as for real z. Since any path of integration between any two complex numbers z_1 and z_2 is approximated arbitrarily closely by a succession of short constant-ρ and constant-ϕ segments, it follows generally that

$$\int_{z_1}^{z_2} z^{n-1}\,dz = \frac{z_2^n - z_1^n}{n}, \quad n \neq 0. \tag{8.22}$$

The applied mathematician might reasonably ask, "Was (8.22) really worth the trouble? We knew *that* already. It's the same as for real numbers."

Well, we really didn't know it before deriving it, but the point is well taken nevertheless. However, notice the exemption of $n = 0$. Equation (8.22) does not hold in that case. Consider the $n = 0$ integral

$$S_0 = \int_{z_1}^{z_2} \frac{dz}{z}.$$

Following the same steps as before and using (5.7) and (2.35), we find that

$$\int_{\rho_1}^{\rho_2} \frac{dz}{z} = \int_{\rho_1}^{\rho_2} \frac{(d\rho + i\rho\,d\phi)e^{i\phi}}{\rho e^{i\phi}} = \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho} = \ln\frac{\rho_2}{\rho_1}. \tag{8.23}$$

This is always real-valued, but otherwise it brings no surprise. However,

$$\int_{\phi_1}^{\phi_2} \frac{dz}{z} = \int_{\phi_1}^{\phi_2} \frac{(d\rho + i\rho\,d\phi)e^{i\phi}}{\rho e^{i\phi}} = i\int_{\phi_1}^{\phi_2} d\phi = i(\phi_2 - \phi_1). \tag{8.24}$$

The odd thing about this is in what happens when the contour closes a complete loop in the Argand plane about the $z = 0$ pole. In this case, $\phi_2 = \phi_1 + 2\pi$, thus

$$S_0 = i2\pi,$$

even though the start and end points of the integration are the same.

Generalizing, we have that

$$\begin{aligned} \oint (z - z_o)^{n-1}\,dz &= 0, \quad n \neq 0; \\ \oint \frac{dz}{z - z_o} &= i2\pi; \end{aligned} \tag{8.25}$$

where as in § 7.6 the symbol $\oint$ represents integration about a closed contour which ends at the same point it began, and where it is implied that the contour loops positively (counterclockwise, in the direction of increasing ϕ) exactly once about the $z = z_o$ pole.

Notice that the formula's $i2\pi$ does not depend on the precise path of integration, but only on the fact that the path loops once positively about the pole. Notice also that nothing in the derivation of (8.22) actually requires that n be an integer, so one can write

$$\int_{z_1}^{z_2} z^{a-1}\,dz = \frac{z_2^a - z_1^a}{a}, \quad a \neq 0. \tag{8.26}$$

However, (8.25) does not hold in the latter case; its integral comes to zero for nonintegral a only if the contour does not enclose the branch point at $z = z_o$.

For a closed contour *which encloses no pole or other nonanalytic point,* (8.26) has that $\oint z^{a-1}\,dz = 0$, or with the change of variable $z - z_o \leftarrow z$,

$$\oint (z - z_o)^{a-1}\,dz = 0.$$

But because any analytic function can be expanded in the form $f(z) = \sum_k (c_k)(z - z_o)^{a_k - 1}$ (which is just a Taylor series if the a_k happen to be positive integers), this means that

$$\oint f(z)\,dz = 0 \tag{8.27}$$

if $f(z)$ is everywhere analytic within the contour.[10]

[10]The careful reader will observe that (8.27)'s derivation does not explicitly handle

8.6.3 The formula

The combination of (8.25) and (8.27) is powerful. Consider the closed contour integral

$$\oint \frac{f(z)}{z - z_o}\,dz,$$

where the contour encloses no nonanalytic point of $f(z)$ itself but does enclose the pole of $f(z)/(z-z_o)$ at $z = z_o$. If the contour were a tiny circle of infinitesimal radius about the pole, then the integrand would reduce to $f(z_o)/(z-z_o)$; and then per (8.25),

$$\oint \frac{f(z)}{z - z_o}\,dz = i2\pi f(z_o). \tag{8.28}$$

But if the contour were not an infinitesimal circle but rather the larger contour of Fig. 8.2? In this case, if the dashed detour which excludes the pole is taken, then according to (8.27) the resulting integral totals zero; but the two straight integral segments evidently cancel; and similarly as we have just reasoned, the *reverse-directed* integral about the tiny detour circle is $-i2\pi f(z_o)$; so to bring the total integral to zero the integral about the main contour must be $i2\pi f(z_o)$. Thus, (8.28) holds for any positively-directed contour which once encloses a pole and no other nonanalytic point, whether the contour be small or large. Equation (8.28) is *Cauchy's integral formula.*

If the contour encloses multiple poles (§§ 2.11 and 9.6.2), then by the principle of linear superposition (§ 7.3.3),

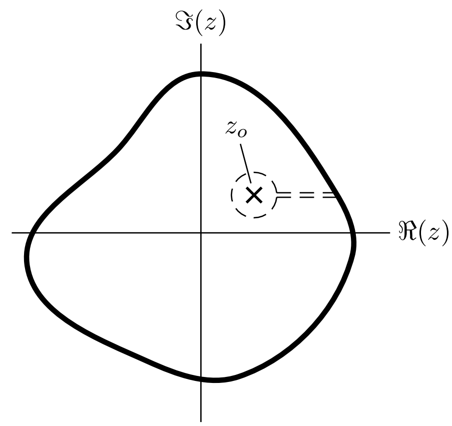
$$\oint \left[f_o(z) + \sum_k \frac{f_k(z)}{z - z_k} \right] dz = i2\pi \sum_k f_k(z_k), \tag{8.29}$$

where the $f_o(z)$ is a *regular part* [16, § 1.1]. The values $f_k(z_k)$, which represent the strengths of the poles, are called *residues.* In words, (8.29) says that an integral about a closed contour in the Argand plane comes to $i2\pi$ times

---

an $f(z)$ represented by a Taylor series with an infinite number of terms and a finite convergence domain (for example, $f(z) = \ln[1 + z]$). However, by § 8.2 one can transpose such a series from z_o to an overlapping convergence domain about z_1. Let the contour's interior be divided into several cells, each of which is small enough to enjoy a single convergence domain. Integrate about each cell. Because the cells share boundaries within the contour's interior, each interior boundary is integrated twice, once in each direction, canceling. The original contour—each piece of which is an exterior boundary of some cell—is integrated once piecewise. This is the basis on which a more rigorous proof is constructed.

Figure 8.2: A Cauchy contour integral.



the sum of the residues of the poles (if any) thus enclosed. (Note however that eqn. 8.29 does not handle branch points. If there is a branch point, the contour must exclude it or the formula will not work.)

As we shall see in § 9.5, whether in the form of (8.28) or of (8.29) Cauchy's integral formula is an extremely useful result.

(References: [15, § 10.6]; [24]; [3, "Cauchy's integral formula," 14:13, 20 April 2006].)

8.7 Taylor series for specific functions

With the general Taylor series formula (8.19), the derivatives of Tables 5.2 and 5.3, and the observation from (4.21) that

$$\frac{d(z^a)}{dz} = az^{a-1},$$

one can calculate Taylor series for many functions. For instance, expanding about $z = 1$,

$$
\begin{aligned}
\ln z\big|_{z=1} &= & \ln z\big|_{z=1} &= 0, \\
\left.\frac{d}{dz}\ln z\right|_{z=1} &= & \left.\frac{1}{z}\right|_{z=1} &= 1, \\
\left.\frac{d^2}{dz^2}\ln z\right|_{z=1} &= & \left.\frac{-1}{z^2}\right|_{z=1} &= -1, \\
\left.\frac{d^3}{dz^3}\ln z\right|_{z=1} &= & \left.\frac{2}{z^3}\right|_{z=1} &= 2, \\
& & &\vdots \\
\left.\frac{d^k}{dz^k}\ln z\right|_{z=1} &= & \left.\frac{-(-)^k(k-1)!}{z^k}\right|_{z=1} &= -(-)^k(k-1)!, \quad k > 0.
\end{aligned}
$$

With these derivatives, the Taylor series about $z = 1$ is

$$
\ln z = \sum_{k=1}^{\infty}\left[-(-)^k(k-1)!\right]\frac{(z-1)^k}{k!} = -\sum_{k=1}^{\infty}\frac{(1-z)^k}{k},
$$

evidently convergent for $|1 - z| < 1$. (And if z lies outside the convergence domain? Several strategies are then possible. One can expand the Taylor series about a different point; but cleverer and easier is to take advantage of some convenient relationship like $\ln w = -\ln[1/w]$, applying the Taylor series given to $z = 1/w$.) Using such Taylor series, one can relatively efficiently calculate actual numerical values for $\ln z$ and many other functions.

Table 8.1 lists Taylor series for a few functions of interest. All the series converge for $|z| < 1$. The $\exp z$, $\sin z$ and $\cos z$ series converge for all complex z. Among the several series, the series for $\arctan z$ is calculated indirectly [22, § 11-7] by way of Table 5.3 and (2.29):

$$
\begin{aligned}
\arctan z &= \int_0^z \frac{1}{1+w^2}\,dw \\
&= \int_0^z \sum_{k=0}^{\infty}(-)^k w^{2k}\,dw \\
&= \sum_{k=0}^{\infty}\frac{(-)^k z^{2k+1}}{2k+1}.
\end{aligned}
$$

Table 8.1: Taylor series.

$$
\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} \left(\left. \frac{d^k f}{dz^k} \right|_{z=z_o} \right) \prod_{j=1}^{k} \frac{z - z_o}{j} \\
(1+z)^{a-1} &= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \left(\frac{a}{j} - 1 \right) z \\
\exp z &= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{z}{j} = \sum_{k=0}^{\infty} \frac{z^k}{k!} \\
\sin z &= (z) \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{-z^2}{(2j)(2j+1)} \\
\cos z &= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{-z^2}{(2j-1)(2j)} \\
\ln(1+z) &= -\sum_{k=1}^{\infty} \frac{(-z)^k}{k} \\
\arctan z &= \sum_{k=0}^{\infty} \frac{(-)^k z^{2k+1}}{2k+1} = (z) \sum_{k=0}^{\infty} \frac{(-z^2)^k}{2k+1}.
\end{aligned}
$$

8.8 Bounds

One naturally cannot actually sum a Taylor series to an infinite number of terms. One must add some finite number of terms, then quit—which raises the question: how can one know that one has added enough terms, that the remaining terms are sufficiently insignificant? How can one set bounds on the infinite sum?

Some series alternate sign. For these it is easy. For example, from Table 8.1,

$$\ln \frac{3}{2} = \frac{1}{(1)(2^1)} - \frac{1}{(2)(2^2)} + \frac{1}{(3)(2^3)} - \frac{1}{(4)(2^4)} + \cdots$$

Each term is smaller in magnitude than the last, so the true value of $\ln(3/2)$ necessarily lies between the sum of the series to $n-1$ terms and the sum to n terms. The last two partial sums bound the result. For instance,

$$\begin{aligned} S_3 - \frac{1}{(4)(2^4)} < \ln \frac{3}{2} &< S_3, \\ S_3 \equiv \frac{1}{(1)(2^1)} - \frac{1}{(2)(2^2)} &+ \frac{1}{(3)(2^3)}. \end{aligned}$$

Other series however do not alternate sign. For example,

$$\begin{aligned} -\ln \frac{1}{2} &= S_4 + R, \\ S_4 &= \frac{1}{(1)(2^1)} + \frac{1}{(2)(2^2)} + \frac{1}{(3)(2^3)} + \frac{1}{(4)(2^4)}, \\ R &\equiv \frac{1}{(5)(2^5)} + \frac{1}{(6)(2^6)} + \cdots \end{aligned}$$

The basic technique in such a case is to find a replacement series (or integral) R' which one can collapse analytically, each of whose terms equals or exceeds in magnitude the corresponding term of R. For the example, one might choose

$$R' = \frac{1}{5} \sum_{k=5}^{\infty} \frac{1}{2^k} = \frac{2}{(5)(2^5)},$$

wherein (2.30) had been used to collapse the summation. Then,

$$S_4 < -\ln \frac{1}{2} < S_4 + R'.$$

For real $0 \le x < 1$ generally,

$$\begin{aligned} S_n &< -\ln(1-x) < S_n + R', \\ S_n &\equiv \sum_{k=1}^{n} \frac{x^k}{k}, \\ R' &\equiv \sum_{k=n+1}^{\infty} \frac{x^k}{n+1} = \frac{x^{n+1}}{(n+1)(1-x)}. \end{aligned}$$

Many variations and refinements are possible, but that is the basic technique: to add several terms of the series to establish a lower bound, then to overestimate the remainder of the series to establish an upper bound. The overestimate *majorizes* the remainder of the series. Notice that the R' in the example is a fairly small number, and that it would have been a lot smaller yet had we included a few more terms in S_n (for instance, $n = \mathrm{0x40}$ would have bound $-\ln[1/2]$ tighter than the limit of a computer's typical `double`-type floating-point accuracy). The technique usually works well in practice for this reason.

Where complex numbers are involved, (3.20) can help.

The same technique can be used to prove that a series does not converge at all. For example,

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

does not converge because

$$\frac{1}{k} > \int_{k}^{k+1} \frac{d\tau}{\tau};$$

hence,

$$\sum_{k=1}^{\infty} \frac{1}{k} > \sum_{k=1}^{\infty} \int_{k}^{k+1} \frac{d\tau}{\tau} = \int_{1}^{\infty} \frac{d\tau}{\tau} = \ln \infty.$$

8.9 Calculating 2π

The Taylor series for $\arctan z$ in Table 8.1 implies a neat way of calculating the constant 2π. We already know that $\tan 2\pi/8 = 1$, or in other words that

$$\arctan 1 = \frac{2\pi}{8}.$$

Applying the Taylor series, we have that

$$2\pi = 8 \sum_{k=0}^{\infty} \frac{(-)^k}{2k+1}. \tag{8.30}$$

The series (8.30) is simple but converges extremely slowly. Much faster convergence is given by angles smaller than $2\pi/8$. For example, from Table 3.2,

$$\arctan \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{2\pi}{\mathrm{0x18}}.$$

Applying the Taylor series at this angle, we have that

$$2\pi = \mathrm{0x18} \sum_{k=0}^{\infty} \frac{(-)^k}{2k+1} \left(\frac{\sqrt{3}-1}{\sqrt{3}+1} \right)^{2k+1} \approx \mathrm{0x6.487F}. \tag{8.31}$$

8.10 The multidimensional Taylor series

Equation (8.19) has given the Taylor series for functions of a single variable. The idea of the Taylor series does not differ where there are two or more independent variables, only the details are a little more complicated. For example, consider the function $f(z_1, z_2) = z_1^2 + z_1 z_2 + 2z_2$, which has terms z_1^2 and $2z_2$—these we understand—but also has the cross-term $z_1 z_2$ for which the relevant derivative is the cross-derivative $\partial^2 f / \partial z_1 \, \partial z_2$. Where two or more independent variables are involved, one must account for the cross-derivatives, too.

With this idea in mind, the multidimensional Taylor series is

$$f(\mathbf{z}) = \sum_{\mathbf{k}} \left(\left. \frac{\partial^{\mathbf{k}} f}{\partial \mathbf{z}^{\mathbf{k}}} \right|_{\mathbf{z}=\mathbf{z}_o} \right) \frac{(\mathbf{z}-\mathbf{z}_o)^{\mathbf{k}}}{\mathbf{k}!}. \tag{8.32}$$

Well, that's neat. What does it mean?

- The $\mathbf{z}$ is a *vector*[11] incorporating the several independent variables $z_1, z_2, \ldots, z_N$.

[11] In this generalized sense of the word, a *vector* is an ordered set of N elements. The geometrical vector $\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ of § 3.3, then, is a vector with $N = 3$, $v_1 = x$, $v_2 = y$ and $v_3 = z$.

- The $\mathbf{k}$ is a nonnegative integer vector of N counters—$k_1, k_2, \ldots, k_N$—one for each of the independent variables. Each of the k_n runs independently from 0 to ∞, and every permutation is possible. For example, if $N = 2$ then

$$\begin{aligned}\mathbf{k} &= (k_1, k_2)\\ &= (0,0),(0,1),(0,2),(0,3),\ldots;\\ &\ (1,0),(1,1),(1,2),(1,3),\ldots;\\ &\ (2,0),(2,1),(2,2),(2,3),\ldots;\\ &\ (3,0),(3,1),(3,2),(3,3),\ldots;\\ &\ \ldots\end{aligned}$$

- The $\partial^{\mathbf{k}} f/\partial \mathbf{z}^{\mathbf{k}}$ represents the $\mathbf{k}$th cross-derivative of $f(\mathbf{z})$, meaning that

$$\frac{\partial^{\mathbf{k}} f}{\partial \mathbf{z}^{\mathbf{k}}} \equiv \left(\prod_{n=1}^{N} \frac{\partial^{k_n}}{(\partial z_n)^{k_n}} \right) f.$$

- The $(\mathbf{z} - \mathbf{z}_o)^{\mathbf{k}}$ represents

$$(\mathbf{z} - \mathbf{z}_o)^{\mathbf{k}} \equiv \prod_{n=1}^{N} (z_n - z_{on})^{k_n}.$$

- The $\mathbf{k}!$ represents

$$\mathbf{k}! \equiv \prod_{n=1}^{N} k_n!.$$

With these definitions, the multidimensional Taylor series (8.32) yields all the right derivatives and cross-derivatives at the expansion point $\mathbf{z} = \mathbf{z}_o$. Thus within some convergence domain about $\mathbf{z} = \mathbf{z}_o$, the multidimensional Taylor series (8.32) represents a function $f(\mathbf{z})$ as accurately as the simple Taylor series (8.19) represents a function $f(z)$, and for the same reason.

Chapter 9

Integration techniques

Equation (4.19) implies a general technique for calculating a derivative symbolically. Its counterpart (7.1), unfortunately, implies a general technique only for calculating an integral *numerically*—and even for this purpose it is imperfect; for, when it comes to adding an infinite number of infinitesimal elements, how is one actually to do the sum?

It turns out that there is no one general answer to this question. Some functions are best integrated by one technique, some by another; it is hard to guess in advance which technique might work best. The calculation of integrals is difficult, even frustrating, but also creative and edifying. It is a useful mathematical art.

There are many ways to solve an integral. This chapter introduces some of the more common ones.

9.1 Integration by antiderivative

The simplest way to solve an integral is just to look at it, recognizing its integrand to be the derivative of something already known:[1]

$$\int_a^z \frac{df}{d\tau}\,d\tau = f(\tau)|_a^z. \tag{9.1}$$

For instance,

$$\int_1^x \frac{1}{\tau}\,d\tau = \ln\tau|_1^x = \ln x.$$

One merely looks at the integrand $1/\tau$, recognizing it to be the derivative of $\ln\tau$; then directly writes down the solution $\ln\tau|_1^x$. Refer to § 7.2.

[1] The notation $f(\tau)|_a^z$ or $[f(\tau)]_a^z$ means $f(z) - f(a)$.

The technique by itself is pretty limited. However, the frequent object of other integration techniques is to transform an integral into a form to which this basic technique can be applied.

Besides the essential

$$\frac{d}{dz}\left(\frac{z^a}{a}\right) = z^{a-1}, \tag{9.2}$$

Tables 7.1, 5.2 and 5.3 provide several further good derivatives this antiderivative technique can use.

One particular, nonobvious, useful variation on the antiderivative technique seems worth calling out specially here. If $z = \rho e^{i\phi}$, then (8.23) and (8.24) have that

$$\int_{z_1}^{z_2} \frac{dz}{z} = \ln\frac{\rho_2}{\rho_1} + i(\phi_2 - \phi_1). \tag{9.3}$$

This helps, for example, when z_1 and z_2 are real but negative numbers.

9.2 Integration by substitution

Consider the integral

$$S = \int_{x_1}^{x_2} \frac{x\,dx}{1+x^2}.$$

This integral is not in a form one immediately recognizes. However, with the change of variable

$$u \leftarrow 1 + x^2,$$

whose differential is (by successive steps)

$$\begin{aligned} d(u) &= d(1+x^2), \\ du &= 2x\,dx, \end{aligned}$$

the integral is

$$\begin{aligned}
S &= \int_{x=x_1}^{x_2} \frac{x\,dx}{u} \\
&= \int_{x=x_1}^{x_2} \frac{2x\,dx}{2u} \\
&= \int_{u=1+x_1^2}^{1+x_2^2} \frac{du}{2u} \\
&= \left.\frac{1}{2}\ln u\right|_{u=1+x_1^2}^{1+x_2^2} \\
&= \frac{1}{2}\ln\frac{1+x_2^2}{1+x_1^2}.
\end{aligned}$$

To check that the result is correct, per § 7.5 we can take the derivative of the final expression with respect to x_2:

$$\begin{aligned}
\left.\frac{\partial}{\partial x_2}\frac{1}{2}\ln\frac{1+x_2^2}{1+x_1^2}\right|_{x_2=x} &= \left[\frac{1}{2}\frac{\partial}{\partial x_2}\left\{\ln\left(1+x_2^2\right)-\ln\left(1+x_1^2\right)\right\}\right]_{x_2=x} \\
&= \frac{x}{1+x^2},
\end{aligned}$$

which indeed has the form of the integrand we started with, confirming the result.

The technique used to solve the integral is *integration by substitution.* It does not solve all integrals but it does solve many; either alone or in combination with other techniques.

9.3 Integration by parts

Integration by parts is a curious but very broadly applicable technique which begins with the derivative product rule (4.25),

$$d(uv) = u\,dv + v\,du,$$

where $u(\tau)$ and $v(\tau)$ are functions of an independent variable τ. Reordering terms,

$$u\,dv = d(uv) - v\,du.$$

Integrating,

$$\int_{\tau=a}^{b} u\,dv = \left. uv\right|_{\tau=a}^{b} - \int_{\tau=a}^{b} v\,du. \tag{9.4}$$

Equation (9.4) is the rule of *integration by parts.*

For an example of the rule's operation, consider the integral

$$S(x) = \int_0^x \tau \cos \alpha\tau \, d\tau.$$

Unsure how to integrate this, we can begin by integrating *part* of it. We can begin by integrating the $\cos \alpha\tau \, d\tau$ part. Letting

$$\begin{aligned} u &\leftarrow \tau, \\ dv &\leftarrow \cos \alpha\tau \, d\tau, \end{aligned}$$

we find that[2]

$$\begin{aligned} du &= d\tau, \\ v &= \frac{\sin \alpha\tau}{\alpha}. \end{aligned}$$

According to (9.4), then,

$$S(x) = \left. \frac{\tau \sin \alpha\tau}{\alpha} \right|_0^x - \int_0^x \frac{\sin \alpha\tau}{\alpha} \, d\tau = \frac{x}{\alpha} \sin \alpha x + \cos \alpha x - 1.$$

Integration by parts is a powerful technique, but one should understand clearly what it does and does not do. It does not integrate each part of an integral separately. It isn't that simple. The reward for integrating the part dv is only a new integral $\int v \, du$, which may or may not be easier to handle than the original $\int u \, dv$. The virtue of the technique lies in that one often can find a part dv which does yield an easier $\int v \, du$. The technique is powerful for this reason.

For another kind of example of the rule's operation, consider the definite integral [16]

$$\Gamma(z) \equiv \int_0^\infty e^{-\tau} \tau^{z-1} \, d\tau, \quad \Re(z) > 0. \tag{9.5}$$

Letting

$$\begin{aligned} u &\leftarrow e^{-\tau}, \\ dv &\leftarrow \tau^{z-1} \, d\tau, \end{aligned}$$

[2]The careful reader will observe that $v = (\sin \alpha\tau)/\alpha + C$ matches the chosen dv for any value of C, not just for $C = 0$. This is true. However, nothing in the integration by parts technique requires us to consider all possible v. Any convenient v suffices. In this case, we choose $v = (\sin \alpha\tau)/\alpha$.

we evidently have that

$$\begin{aligned} du &= -e^{-\tau}\,d\tau, \\ v &= \frac{\tau^z}{z}. \end{aligned}$$

Substituting these according to (9.4) into (9.5) yields

$$\begin{aligned} \Gamma(z) &= \left[e^{-\tau}\frac{\tau^z}{z}\right]_{\tau=0}^{\infty} - \int_0^\infty \left(\frac{\tau^z}{z}\right)\left(-e^{-\tau}\,d\tau\right) \\ &= [0-0] + \int_0^\infty \frac{\tau^z}{z}e^{-\tau}\,d\tau \\ &= \frac{\Gamma(z+1)}{z}. \end{aligned}$$

When written

$$\Gamma(z+1) = z\Gamma(z), \tag{9.6}$$

this is an interesting result. Since per (9.5)

$$\Gamma(1) = \int_0^\infty e^{-\tau}\,d\tau = \left[-e^{-\tau}\right]_0^\infty = 1,$$

it follows by induction on (9.6) that

$$(n-1)! = \Gamma(n). \tag{9.7}$$

Thus (9.5), called the *Gamma function,* can be taken as an extended definition of the factorial $(\cdot)!$ for all z, $\Re(z) > 0$. Integration by parts has made this finding possible.

9.4 Integration by unknown coefficients

One of the more powerful integration techniques is relatively inelegant, yet it easily cracks some integrals the other techniques have trouble with. The technique is the *method of unknown coefficients,* and it is based on the antiderivative (9.1) plus intelligent guessing. It is best illustrated by example.

Consider the integral (which arises in probability theory)

$$S(x) = \int_0^x e^{-(\rho/\sigma)^2/2}\rho\,d\rho. \tag{9.8}$$

If one does not know how to solve the integral in a more elegant way, one can *guess* a likely-seeming antiderivative form, such as

$$e^{-(\rho/\sigma)^2/2}\rho = \frac{d}{d\rho} a e^{-(\rho/\sigma)^2/2},$$

where the a is an *unknown coefficient.* Having guessed, one has no guarantee that the guess is right, but see: if the guess *were* right, then the antiderivative would have the form

$$\begin{aligned} e^{-(\rho/\sigma)^2/2}\rho &= \frac{d}{d\rho} a e^{-(\rho/\sigma)^2/2} \\ &= -\frac{a\rho}{\sigma^2} e^{-(\rho/\sigma)^2/2}, \end{aligned}$$

implying that

$$a = -\sigma^2$$

(evidently the guess is right, after all). Using this value for a, one can write the specific antiderivative

$$e^{-(\rho/\sigma)^2/2}\rho = \frac{d}{d\rho}\left[-\sigma^2 e^{-(\rho/\sigma)^2/2}\right],$$

with which one can solve the integral, concluding that

$$S(x) = \left[-\sigma^2 e^{-(\rho/\sigma)^2/2}\right]_0^x = \left(\sigma^2\right)\left(1 - e^{-(x/\sigma)^2/2}\right). \tag{9.9}$$

The same technique solves differential equations, too. Consider for example the differential equation

$$dx = (Ix - P)\,dt, \quad x|_{t=0} = x_o, \; x|_{t=T} = 0, \tag{9.10}$$

which conceptually represents[3] the changing balance x of a bank loan account over time t, where I is the loan's interest rate and P is the borrower's payment rate. If it is desired to find the correct payment rate P which pays the loan off in the time T, then (perhaps after some bad guesses) we guess the form

$$x(t) = Ae^{\alpha t} + B,$$

where α, A and B are unknown coefficients. The guess' derivative is

$$dx = \alpha A e^{\alpha t}\,dt.$$

[3] Real banks (in the author's country, at least) by law or custom actually use a needlessly more complicated formula—and not only more complicated, but mathematically slightly incorrect, too.

Substituting the last two equations into (9.10) and dividing by dt yields

$$\alpha A e^{\alpha t} = I A e^{\alpha t} + IB - P,$$

which at least is satisfied if both of the equations

$$\begin{aligned} \alpha A e^{\alpha t} &= I A e^{\alpha t}, \\ 0 &= IB - P, \end{aligned}$$

are satisfied. Evidently good choices for α and B, then, are

$$\begin{aligned} \alpha &= I, \\ B &= \frac{P}{I}. \end{aligned}$$

Substituting these coefficients into the $x(t)$ equation above yields the general solution

$$x(t) = Ae^{It} + \frac{P}{I} \tag{9.11}$$

to (9.10). The constants A and P, we establish by applying the given *boundary conditions* $x|_{t=0} = x_o$ and $x|_{t=T} = 0$. For the former condition, (9.11) is

$$x_o = Ae^{(I)(0)} + \frac{P}{I} = A + \frac{P}{I};$$

and for the latter condition,

$$0 = Ae^{IT} + \frac{P}{I}.$$

Solving the last two equations simultaneously, we have that

$$\begin{aligned} A &= \frac{-e^{-IT} x_o}{1 - e^{-IT}}, \\ P &= \frac{I x_o}{1 - e^{-IT}}. \end{aligned} \tag{9.12}$$

Applying these to the general solution (9.11) yields the specific solution

$$x(t) = \frac{x_o}{1 - e^{-IT}} \left(1 - e^{(I)(t-T)}\right) \tag{9.13}$$

to (9.10) meeting the boundary conditions, with the payment rate P required of the borrower given by (9.12).

The virtue of the method of unknown coefficients lies in that it permits one to try an entire family of candidate solutions at once, with the family members distinguished by the values of the coefficients. If a solution exists anywhere in the family, the method usually finds it.

The method of unknown coefficients is an elephant. Slightly inelegant the method may be, but it is pretty powerful, too—and it has surprise value (for some reason people seem not to expect it). Such are the kinds of problems the method can solve.

9.5 Integration by closed contour

We pass now from the elephant to the falcon, from the inelegant to the sublime. Consider the integral [16, § 1.2]

$$S = \int_0^\infty \frac{\tau^a}{\tau+1}\,d\tau, \quad -1 < a < 0.$$

This is a hard integral. No obvious substitution, no evident factoring into parts, seems to solve the integral; but there is a way. The integrand has a pole at $\tau = -1$. Observing that τ is only a dummy integration variable, if one writes the same integral using the complex variable z in place of the real variable τ, then Cauchy's integral formula (8.28) has that integrating once counterclockwise about a closed complex contour, with the contour enclosing the pole at $z = -1$ but shutting out the branch point at $z = 0$, yields

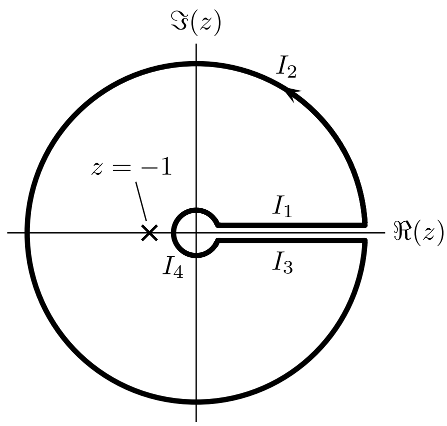
$$I = \oint \frac{z^a}{z+1}\,dz = i2\pi z^a|_{z=-1} = i2\pi\left(e^{i2\pi/2}\right)^a = i2\pi e^{i2\pi a/2}.$$

The trouble, of course, is that the integral S does not go about a closed complex contour. One can however construct a closed complex contour I of which S is a part, as in Fig 9.1. If the outer circle in the figure is of infinite radius and the inner, of infinitesimal, then the closed contour I is composed of the four parts

$$\begin{aligned} I &= I_1 + I_2 + I_3 + I_4 \\ &= (I_1 + I_3) + I_2 + I_4. \end{aligned}$$

The figure tempts one to make the mistake of writing that $I_1 = S = -I_3$, but besides being incorrect this defeats the purpose of the closed contour technique. More subtlety is needed. One must take care to interpret the four parts correctly. The integrand $z^a/(z+1)$ is multiple-valued; so, in fact, the two parts $I_1 + I_3 \neq 0$ do not cancel. The integrand has a branch

Figure 9.1: Integration by closed contour.



point at $z = 0$, which, in passing from I_3 through I_4 to I_1, the contour has circled. Even though z itself takes on the same values along I_3 as along I_1, the multiple-valued integrand $z^a/(z+1)$ does not. Indeed,

$$
\begin{aligned}
I_1 &= \int_0^\infty \frac{(\rho e^{i0})^a}{(\rho e^{i0}) + 1}\, d\rho &&= \int_0^\infty \frac{\rho^a}{\rho + 1}\, d\rho &&= S, \\
-I_3 &= \int_0^\infty \frac{(\rho e^{i2\pi})^a}{(\rho e^{i2\pi}) + 1}\, d\rho &&= e^{i2\pi a} \int_0^\infty \frac{\rho^a}{\rho + 1}\, d\rho &&= e^{i2\pi a} S.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I &= I_1 + I_2 + I_3 + I_4 \\
&= (I_1 + I_3) + I_2 + I_4 \\
&= (1 - e^{i2\pi a})S + \lim_{\rho\to\infty} \int_{\phi=0}^{2\pi} \frac{z^a}{z+1}\, dz - \lim_{\rho\to 0} \int_{\phi=0}^{2\pi} \frac{z^a}{z+1}\, dz \\
&= (1 - e^{i2\pi a})S + \lim_{\rho\to\infty} \int_{\phi=0}^{2\pi} z^{a-1}\, dz - \lim_{\rho\to 0} \int_{\phi=0}^{2\pi} z^a\, dz \\
&= (1 - e^{i2\pi a})S + \lim_{\rho\to\infty} \left.\frac{z^a}{a}\right|_{\phi=0}^{2\pi} - \lim_{\rho\to 0} \left.\frac{z^{a+1}}{a+1}\right|_{\phi=0}^{2\pi}.
\end{aligned}
$$

Since $a < 0$, the first limit vanishes; and because $a > -1$, the second limit vanishes, too, leaving

$$I = (1 - e^{i2\pi a})S.$$

But by Cauchy's integral formula we have already found an expression for I. Substituting this expression into the last equation yields, by successive steps,

$$\begin{aligned} i2\pi e^{i2\pi a/2} &= (1 - e^{i2\pi a})S, \\ S &= \frac{i2\pi e^{i2\pi a/2}}{1 - e^{i2\pi a}}, \\ S &= \frac{i2\pi}{e^{-i2\pi a/2} - e^{i2\pi a/2}}, \\ S &= -\frac{2\pi/2}{\sin(2\pi a/2)}. \end{aligned}$$

That is,

$$\int_0^\infty \frac{\tau^a}{\tau + 1}\, d\tau = -\frac{2\pi/2}{\sin(2\pi a/2)}, \quad -1 < a < 0, \tag{9.14}$$

an astonishing result.[4]

The foregoing technique, *integration by closed contour,* is found in practice to solve many integrals which other techniques find almost impossible to crack. The key to making the technique work lies in choosing a contour whose individual segments one knows how to treat. The robustness of the technique lies in that any contour of any shape will work, so long as the contour encloses appropriate poles in the Argand domain plane while shutting branch points out.

9.6 Integration by partial-fraction expansion

This section treats integration by partial-fraction expansion. It introduces the expansion itself first. [19, Appendix F][15, §§ 2.7 and 10.12]

9.6.1 Partial-fraction expansion

Consider the function

$$f(z) = \frac{-4}{z - 1} + \frac{5}{z - 2}.$$

[4]So astonishing is the result, that one is unlikely to believe it at first encounter. However, straightforward (though computationally highly inefficient) numerical integration per (7.1) confirms the result, as the interested reader and his computer can check. Such results vindicate the effort we have spent in deriving Cauchy's integral formula (8.28).

Combining the two fractions over a common denominator[5] yields

$$f(z) = \frac{z+3}{(z-1)(z-2)}.$$

Of the two forms, the former is probably the more amenable. For example,

$$\begin{aligned}\int_{-1}^{0} f(\tau)\,d\tau &= \int_{-1}^{0} \frac{-4}{\tau-1}\,d\tau + \int_{-1}^{0} \frac{5}{\tau-2}\,d\tau \\ &= \left[-4\ln(1-\tau) + 5\ln(2-\tau)\right]_{-1}^{0}.\end{aligned}$$

The trouble is that one is not always given the function in the amenable form.

Given a *rational function*

$$f(z) = \frac{\sum_{k=0}^{N} b_k z^k}{\prod_{j=1}^{N}(z-\alpha_j)}, \tag{9.15}$$

the *partial-fraction expansion* has the form

$$f(z) = \sum_{k=1}^{N} \frac{A_k}{z-\alpha_k}, \tag{9.16}$$

where multiplying each fraction of (9.16) by

$$\frac{\left[\prod_{j=1}^{N}(z-\alpha_j)\right]/(z-\alpha_k)}{\left[\prod_{j=1}^{N}(z-\alpha_j)\right]/(z-\alpha_k)}$$

puts the several fractions over a common denominator, yielding (9.15). Dividing (9.15) by (9.16) gives the ratio

$$1 = \frac{\sum_{k=0}^{N} b_k z^k}{\prod_{j=1}^{N}(z-\alpha_j)} \Bigg/ \sum_{k=1}^{N} \frac{A_k}{z-\alpha_k}.$$

In the immediate neighborhood of $z = \alpha_m$, the mth term $A_m/(z-\alpha_m)$ dominates the summation of (9.16). Hence,

$$1 = \lim_{z\to\alpha_m} \frac{\sum_{k=0}^{N} b_k z^k}{\prod_{j=1}^{N}(z-\alpha_j)} \Bigg/ \frac{A_m}{z-\alpha_m}.$$

[5]Terminology (you probably knew this already): A *fraction* is the ratio of two numbers or expressions B/A. In the fraction, B is the *numerator* and A is the *denominator*. The *quotient* is $Q = B/A$.

Rearranging factors, we have that

$$A_m = \left. \frac{\sum_{k=0}^{N} b_k z^k}{\left[\prod_{j=1}^{N}(z-\alpha_j)\right]/(z-\alpha_m)} \right|_{z=\alpha_m}, \tag{9.17}$$

where A_m is called the *residue* of $f(z)$ at the pole $z = \alpha_m$. Equations (9.16) and (9.17) together give the partial-fraction expansion of (9.15)'s rational function $f(z)$.

9.6.2 Multiple poles

The weakness of the partial-fraction expansion of § 9.6.1 is that it cannot directly handle multiple poles. That is, if $\alpha_i = \alpha_j$, $i \neq j$, then the residue formula (9.17) finds an uncanceled pole remaining in its denominator and thus fails for $A_i = A_j$ (it still works for the other A_m). The conventional way to expand a fraction with multiple poles is presented in the references given at the head of the section and in other books; but because at least to this writer that way does not lend much applied insight, the present section treats the matter in a different way.

Consider the function

$$g(z) = \sum_{k=0}^{N-1} \frac{Ce^{i2\pi k/N}}{z - \epsilon e^{i2\pi k/N}}, \quad N > 1, \ 0 < \epsilon \ll 1, \tag{9.18}$$

where C is a real-valued constant. This function evidently has a small circle of poles in the Argand plane at $\alpha_k = \epsilon e^{i2\pi k/N}$. Factoring,

$$g(z) = \frac{C}{z} \sum_{k=0}^{N-1} \frac{e^{i2\pi k/N}}{1 - \epsilon e^{i2\pi k/N}/z}.$$

Using (2.30) to expand the fraction,

$$\begin{aligned} g(z) &= \frac{C}{z} \sum_{k=0}^{N-1} \left[e^{i2\pi k/N} \sum_{j=0}^{\infty} \left(\frac{\epsilon e^{i2\pi k/N}}{z} \right)^j \right] \\ &= C \sum_{k=0}^{N-1} \sum_{j=1}^{\infty} \frac{\epsilon^{j-1} e^{i2\pi jk/N}}{z^j} \\ &= C \sum_{j=1}^{\infty} \frac{\epsilon^{j-1}}{z^j} \sum_{k=0}^{N-1} \left(e^{i2\pi j/N} \right)^k. \end{aligned}$$

But[6]

$$\sum_{k=0}^{N-1}\left(e^{i2\pi j/N}\right)^k = \begin{cases} N & \text{if } j = nN, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$g(z) = NC\sum_{j=1}^{\infty}\frac{\epsilon^{jN-1}}{z^{jN}}.$$

For $|z| \gg \epsilon$—that is, except in the immediate neighborhood of the small circle of poles—the first term of the summation dominates. Hence,

$$g(z) \approx NC\frac{\epsilon^{N-1}}{z^N}, \quad |z| \gg \epsilon.$$

If strategically we choose

$$C = \frac{1}{N\epsilon^{N-1}},$$

then

$$g(z) \approx \frac{1}{z^N}, \quad |z| \gg \epsilon.$$

But given the chosen value of C, (9.18) is

$$g(z) = \frac{1}{N\epsilon^{N-1}}\sum_{k=0}^{N-1}\frac{e^{i2\pi k/N}}{z - \epsilon e^{i2\pi k/N}}, \quad N > 1, \ 0 < \epsilon \ll 1.$$

Joining the last two equations together, changing $z - z_o \leftarrow z$, and writing more formally, we have that

$$\frac{1}{(z - z_o)^N} = \lim_{\epsilon \to 0}\frac{1}{N\epsilon^{N-1}}\sum_{k=0}^{N-1}\frac{e^{i2\pi k/N}}{(z - z_o) - \epsilon e^{i2\pi k/N}}, \quad N > 1. \tag{9.19}$$

Equation (9.19) lets one replace an N-fold pole with a small circle of ordinary poles, which per § 9.6.1 we already know how to handle. Notice incidentally that $1/N\epsilon^{N-1}$ is a large number not a small. The poles are close together but very strong.

[6] If you don't see why, then for $N = 8$ and $j = 3$ plot the several $(e^{i2\pi j/N})^k$ in the Argand plane. Do the same for $j = 2$ then $j = 8$. Only in the $j = 8$ case do the terms add coherently; in the other cases they cancel.

This effect—reinforcing when $j = nN$, canceling otherwise—is a classic manifestation of *Parseval's principle.*

An example to illustrate the technique, separating a double pole:

$$
\begin{aligned}
f(z) &= \frac{z^2+2z+9}{(z-1)^2(z+2)} \\
&= \lim_{\epsilon\to 0} \frac{z^2+2z+9}{(z-[1+\epsilon e^{i2\pi(0)/2}])(z-[1+\epsilon e^{i2\pi(1)/2}])(z+2)} \\
&= \lim_{\epsilon\to 0} \frac{z^2+2z+9}{(z-[1+\epsilon])(z-[1-\epsilon])(z+2)} \\
&= \lim_{\epsilon\to 0} \left\{ \left(\frac{1}{z-[1+\epsilon]}\right)\left[\frac{z^2+2z+9}{(z-[1-\epsilon])(z+2)}\right]_{z=1+\epsilon} \right. \\
&\qquad + \left(\frac{1}{z-[1-\epsilon]}\right)\left[\frac{z^2+2z+9}{(z-[1+\epsilon])(z+2)}\right]_{z=1-\epsilon} \\
&\qquad \left. + \left(\frac{1}{z+2}\right)\left[\frac{z^2+2z+9}{(z-[1+\epsilon])(z-[1-\epsilon])}\right]_{z=-2} \right\} \\
&= \lim_{\epsilon\to 0} \left\{ \left(\frac{1}{z-[1+\epsilon]}\right)\frac{\mathrm{0xC}}{6\epsilon} + \left(\frac{1}{z-[1-\epsilon]}\right)\frac{\mathrm{0xC}}{-6\epsilon} + \left(\frac{1}{z+2}\right)\frac{9}{9} \right\} \\
&= \lim_{\epsilon\to 0} \left\{ \frac{2/\epsilon}{z-[1+\epsilon]} + \frac{-2/\epsilon}{z-[1-\epsilon]} + \frac{1}{z+2} \right\}.
\end{aligned}
$$

9.6.3 Integrating rational functions

If one can find the poles of a rational function of the form (9.15), then one can use (9.16) and (9.17)—and, if needed, (9.19)—to expand the function into a sum of partial fractions, each of which one can integrate individually.

Continuing the example of § 9.6.2, for $0 \le x < 1$,

$$
\begin{aligned}
\int_0^x f(\tau)\,d\tau &= \int_0^x \frac{\tau^2+2\tau+9}{(\tau-1)^2(\tau+2)}\,d\tau \\
&= \lim_{\epsilon\to 0}\int_0^x \left\{ \frac{2/\epsilon}{\tau-[1+\epsilon]} + \frac{-2/\epsilon}{\tau-[1-\epsilon]} + \frac{1}{\tau+2} \right\} d\tau \\
&= \lim_{\epsilon\to 0}\left\{ \frac{2}{\epsilon}\ln([1+\epsilon]-\tau) - \frac{2}{\epsilon}\ln([1-\epsilon]-\tau) + \ln(\tau+2) \right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{ \frac{2}{\epsilon}\ln\left(\frac{[1+\epsilon]-\tau}{[1-\epsilon]-\tau}\right) + \ln(\tau+2) \right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{ \frac{2}{\epsilon}\ln\left(\frac{[1-\tau]+\epsilon}{[1-\tau]-\epsilon}\right) + \ln(\tau+2) \right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{ \frac{2}{\epsilon}\ln\left(1+\frac{2\epsilon}{1-\tau}\right) + \ln(\tau+2) \right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{ \frac{2}{\epsilon}\left(\frac{2\epsilon}{1-\tau}\right) + \ln(\tau+2) \right\}_0^x \\
&= \left\{ \frac{4}{1-\tau} + \ln(\tau+2) \right\}_0^x \\
&= \frac{4}{1-x} - 4 + \ln\left(\frac{x+2}{2}\right).
\end{aligned}
$$

To check that the result is correct, we can take the derivative of the final expression:

$$
\begin{aligned}
\left[\frac{d}{dx}\left\{\frac{4}{1-x} - 4 + \ln\left(\frac{x+2}{2}\right)\right\}\right]_{x=\tau} &= \frac{4}{(1-\tau)^2} + \frac{1}{\tau+2} \\
&= \frac{\tau^2+2\tau+9}{(\tau-1)^2(\tau+2)},
\end{aligned}
$$

which indeed has the form of the integrand we started with, confirming the result. (Notice incidentally how much easier it is symbolically to differentiate than to integrate![7])

[7]Paradoxically, *numerically* it is usually easier to integrate than to differentiate, as § 7.5 has observed. Numerical differentiation is prone to big errors due to sample noise. Numerical integration is basically just repeated addition, which tends to average any sample noise out.

9.7 Integration by Taylor series

With sufficient cleverness the techniques of the foregoing sections solve many, many integrals. But not all. When all else fails, as sometimes it does, the Taylor series of Ch. 8 and the antiderivative of § 9.1 together offer a concise, practical way to integrate some functions, at the price of losing the functions' known closed analytic forms. For example,

$$\begin{aligned}
\int_0^x \exp\left(-\frac{\tau^2}{2}\right) d\tau &= \int_0^x \sum_{k=0}^{\infty} \frac{(-\tau^2/2)^k}{k!}\, d\tau \\
&= \int_0^x \sum_{k=0}^{\infty} \frac{(-)^k \tau^{2k}}{2^k k!}\, d\tau \\
&= \left[\sum_{k=0}^{\infty} \frac{(-)^k \tau^{2k+1}}{(2k+1)2^k k!} \right]_0^x \\
&= \sum_{k=0}^{\infty} \frac{(-)^k x^{2k+1}}{(2k+1)2^k k!} = (x) \sum_{k=0}^{\infty} \frac{1}{2k+1} \prod_{j=1}^{k} \frac{-x^2}{2j}.
\end{aligned}$$

The result is no function one recognizes; it is just a series. This is not necessarily bad, however. After all, when a Taylor series from Table 8.1 is used to calculate $\sin z$, then $\sin z$ is just a series, too. The series above converges just as accurately and just as fast.

Sometimes it helps to give the series a name like

$$\operatorname{myf} z \equiv \sum_{k=0}^{\infty} \frac{(-)^k z^{2k+1}}{(2k+1)2^k k!} = (z) \sum_{k=0}^{\infty} \frac{1}{2k+1} \prod_{j=1}^{k} \frac{-z^2}{2j}.$$

Then,

$$\int_0^x \exp\left(-\frac{\tau^2}{2}\right) d\tau = \operatorname{myf} x.$$

The $\operatorname{myf} z$ is no less a function than $\sin z$ is; it's just a function you hadn't heard of before. You can plot the function, or take its derivative

$$\frac{d}{d\tau} \operatorname{myf} \tau = \exp\left(-\frac{\tau^2}{2}\right),$$

or calculate its value, or do with it whatever else one does with functions. It works just the same.

Chapter 10

Cubics and quartics

Under the heat of noonday, between the hard work of the morning and the heavy lifting of the afternoon, one likes to lay down one's burden and rest a spell in the shade. Chapters 2 through 9 have established the applied mathematical foundations upon which coming chapters will build; and chapters 11 through 14, hefting the weighty topic of the matrix, will indeed begin to build on those foundations. But in this short chapter which rests between, we shall refresh ourselves with an interesting but lighter mathematical topic: the topic of cubics and quartics.

The expression

$$z + a_0$$

is a *linear* polynomial, the lone root $z = -a_0$ of which is plain to see. The *quadratic* polynomial

$$z^2 + a_1 z + a_0$$

has of course two roots, which though not plain to see the quadratic formula (2.2) extracts with little effort. So much algebra has been known since antiquity. The roots of higher-order polynomials, the Newton-Raphson iteration (4.29) locates swiftly, but that is an approximate iteration rather than an exact formula like (2.2), and as we have seen in § 4.8 it can occasionally fail to converge. One would prefer an actual formula to extract the roots.

No general formula to extract the roots of the nth-order polynomial seems to be known.[1] However, to extract the roots of the *cubic* and *quartic* polynomials

$$\begin{aligned} z^3 + a_2 z^2 + a_1 z + a_0, \\ z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, \end{aligned}$$

[1] Refer to Ch. 6's footnote 6.

though the ancients never discovered how, formulas do exist. The 16th-century algebraists Ferrari, Vieta, Tartaglia and Cardano have given us the clever technique.

(References: [28, "Cubic equation"]; [28, "Quartic equation"]; [3, "Quartic equation," 00:26, 9 Nov. 2006]; [3, "François Viète," 05:17, 1 Nov. 2006]; [3, "Gerolamo Cardano," 22:35, 31 Oct. 2006]; [26, § 1.5].)

10.1 Vieta's transform

There is a sense to numbers by which 1/2 resembles 2, 1/3 resembles 3, 1/4 resembles 4, and so forth. To capture this sense, one can transform a function $f(z)$ into a function $f(w)$ by the change of variable[2]

$$w + \frac{1}{w} \leftarrow z,$$

or, more generally,

$$w + \frac{w_o^2}{w} \leftarrow z. \tag{10.1}$$

Equation (10.1) is *Vieta's transform.*[3]

For $|w| \gg |w_o|$, $z \approx w$; but as $|w|$ approaches $|w_o|$, this ceases to be true. For $|w| \ll |w_o|$, $z \approx w_o^2/w$. The constant w_o is the *corner value,* in the neighborhood of which w transitions from the one domain to the other. Figure 10.1 plots Vieta's transform for real w in the case $w_o = 1$.

An interesting alternative to Vieta's transform is

$$w \parallel \frac{w_o^2}{w} \leftarrow z, \tag{10.2}$$

which in light of § 6.3 might be named *Vieta's parallel transform.*

Section 10.2 shows how Vieta's transform can be used.

10.2 Cubics

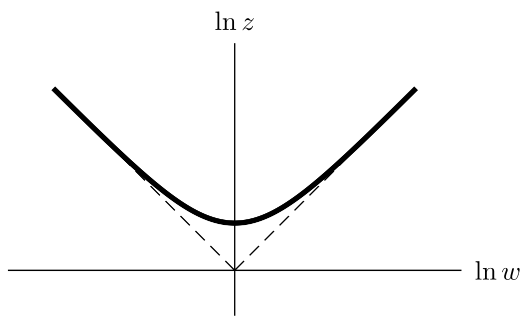
The general cubic polynomial is too hard to extract the roots of directly, so one begins by changing the variable

$$x + h \leftarrow z \tag{10.3}$$

[2]This change of variable broadly recalls the sum-of-exponentials form (5.19) of the $\cosh(\cdot)$ function, inasmuch as $\exp[-\phi] = 1/\exp\phi$.

[3]Also called "Vieta's substitution." [28, "Vieta's substitution"]

Figure 10.1: Vieta's transform (10.1) for $w_o = 1$, plotted logarithmically.



to obtain the polynomial

$$x^3 + (a_2 + 3h)x^2 + (a_1 + 2ha_2 + 3h^2)x + (a_0 + ha_1 + h^2 a_2 + h^3).$$

The choice

$$h \equiv -\frac{a_2}{3} \tag{10.4}$$

casts the polynomial into the improved form

$$x^3 + \left[a_1 - \frac{a_2^2}{3}\right] x + \left[a_0 - \frac{a_1 a_2}{3} + 2\left(\frac{a_2}{3}\right)^3\right],$$

or better yet

$$x^3 - px - q,$$

where

$$\begin{aligned} p &\equiv -a_1 + \frac{a_2^2}{3}, \\ q &\equiv -a_0 + \frac{a_1 a_2}{3} - 2\left(\frac{a_2}{3}\right)^3. \end{aligned} \tag{10.5}$$

The solutions to the equation

$$x^3 = px + q, \tag{10.6}$$

then, are the cubic polynomial's three roots.

So we have struck the $a_2 z^2$ term. That was the easy part; what to do next is not so obvious. If one could strike the px term as well, then the roots would follow immediately, but no very simple substitution like (10.3) achieves this—or rather, such a substitution does achieve it, but at the price of reintroducing an unwanted x^2 or z^2 term. That way is no good. Lacking guidance, one might try many, various substitutions, none of which seems to help much; but after weeks or months of such frustration one might eventually discover Vieta's transform (10.1), with the idea of balancing the equation between offsetting w and $1/w$ terms. This works.

Vieta-transforming (10.6) by the change of variable

$$w + \frac{w_o^2}{w} \leftarrow x \tag{10.7}$$

we get the new equation

$$w^3 + (3w_o^2 - p)w + (3w_o^2 - p)\frac{w_o^2}{w} + \frac{w_o^6}{w^3} = q, \tag{10.8}$$

which invites the choice

$$w_o^2 \equiv \frac{p}{3}, \tag{10.9}$$

reducing (10.8) to read

$$w^3 + \frac{(p/3)^3}{w^3} = q.$$

Multiplying by w^3 and rearranging terms, we have the quadratic equation

$$(w^3)^2 = 2\left(\frac{q}{2}\right)w^3 - \left(\frac{p}{3}\right)^3, \tag{10.10}$$

which by (2.2) we know how to solve.

Vieta's transform has reduced the original cubic to a quadratic.

The careful reader will observe that (10.10) seems to imply six roots, double the three the fundamental theorem of algebra (§ 6.2.2) allows a cubic polynomial to have. We shall return to this point in § 10.3. For the moment, however, we should like to improve the notation by defining[4]

$$\begin{aligned} P &\leftarrow -\frac{p}{3}, \\ Q &\leftarrow +\frac{q}{2}, \end{aligned} \tag{10.11}$$

[4]Why did we not define P and Q so to begin with? Well, before unveiling (10.10), we lacked motivation to do so. To define inscrutable coefficients unnecessarily before the need for them is apparent seems poor applied mathematical style.

Table 10.1: The method to extract the three roots of the general cubic polynomial. (In the definition of w^3, one can choose either sign.)

$$
\begin{aligned}
0 &= z^3 + a_2 z^2 + a_1 z + a_0 \\
P &\equiv \frac{a_1}{3} - \left(\frac{a_2}{3}\right)^2 \\
Q &\equiv \frac{1}{2}\left[-a_0 + 3\left(\frac{a_1}{3}\right)\left(\frac{a_2}{3}\right) - 2\left(\frac{a_2}{3}\right)^3\right] \\
w^3 &\equiv \begin{cases} 2Q & \text{if } P = 0, \\ Q \pm \sqrt{Q^2 + P^3} & \text{otherwise.} \end{cases} \\
x &\equiv \begin{cases} 0 & \text{if } P = 0 \text{ and } Q = 0, \\ w - P/w & \text{otherwise.} \end{cases} \\
z &= x - \frac{a_2}{3}
\end{aligned}
$$

with which (10.6) and (10.10) are written

$$x^3 = 2Q - 3Px, \tag{10.12}$$

$$(w^3)^2 = 2Qw^3 + P^3. \tag{10.13}$$

Table 10.1 summarizes the complete cubic polynomial root extraction method in the revised notation—including a few fine points regarding superfluous roots and edge cases, treated in §§ 10.3 and 10.4 below.

10.3 Superfluous roots

As § 10.2 has observed, the equations of Table 10.1 seem to imply six roots, double the three the fundamental theorem of algebra (§ 6.2.2) allows a cubic polynomial to have. However, what the equations really imply is not six distinct roots but six distinct w. The definition $x \equiv w - P/w$ maps two w to any one x, so in fact the equations imply only three x and thus three roots z. The question then is: of the six w, which three do we really need and which three can we ignore as superfluous?

The six w naturally come in two groups of three: one group of three from the one w^3 and a second from the other. For this reason, we shall guess—and logically it is only a guess—that a single w^3 generates three distinct x and thus (because z differs from x only by a constant offset) all three roots z. If the guess is right, then the second w^3 cannot but yield the same three roots, which means that the second w^3 is superfluous and can safely be overlooked. But is the guess right? Does a single w^3 in fact generate three distinct x?

To prove that it does, let us suppose that it did not. Let us suppose that a single w^3 did generate two w which led to the same x. Letting the symbol w_1 represent the third w, then (since all three w come from the same w^3) the two w are $e^{+i2\pi/3}w_1$ and $e^{-i2\pi/3}w_1$. Because $x \equiv w - P/w$, by successive steps,

$$\begin{aligned}
e^{+i2\pi/3}w_1 - \frac{P}{e^{+i2\pi/3}w_1} &= e^{-i2\pi/3}w_1 - \frac{P}{e^{-i2\pi/3}w_1}, \\
e^{+i2\pi/3}w_1 + \frac{P}{e^{-i2\pi/3}w_1} &= e^{-i2\pi/3}w_1 + \frac{P}{e^{+i2\pi/3}w_1}, \\
e^{+i2\pi/3}\left(w_1 + \frac{P}{w_1}\right) &= e^{-i2\pi/3}\left(w_1 + \frac{P}{w_1}\right),
\end{aligned}$$

which can only be true if

$$w_1^2 = -P.$$

Cubing[5] the last equation,

$$w_1^6 = -P^3;$$

but squaring the table's w^3 definition for $w = w_1$,

$$w_1^6 = 2Q^2 + P^3 \pm 2Q\sqrt{Q^2 + P^3}.$$

Combining the last two on w_1^6,

$$-P^3 = 2Q^2 + P^3 \pm 2Q\sqrt{Q^2 + P^3},$$

or, rearranging terms and halving,

$$Q^2 + P^3 = \mp Q\sqrt{Q^2 + P^3}.$$

Squaring,

$$Q^4 + 2Q^2P^3 + P^6 = Q^4 + Q^2P^3,$$

then canceling offsetting terms and factoring,

$$(P^3)(Q^2 + P^3) = 0.$$

[5]The verb *to cube* in this context means "to raise to the third power," as to change y to y^3, just as the verb *to square* means "to raise to the second power."

The last equation demands rigidly that either $P = 0$ or $P^3 = -Q^2$. Some cubic polynomials do meet the demand—§ 10.4 will treat these and the reader is asked to set them aside for the moment—but most cubic polynomials do not meet it. For most cubic polynomials, then, the contradiction proves false the assumption which gave rise to it. The assumption: that the three x descending from a single w^3 were not distinct. Therefore, provided that $P \neq 0$ and $P^3 \neq -Q^2$, the three x descending from a single w^3 are indeed distinct, as was to be demonstrated.

The conclusion: *either, not both, of the two signs in the table's quadratic solution* $w^3 \equiv Q \pm \sqrt{Q^2 + P^3}$ *demands to be considered.* One can choose either sign; it matters not which.[6] The one sign alone yields all three roots of the general cubic polynomial.

In calculating the three w from w^3, one can apply the Newton-Raphson iteration (4.31), the Taylor series of Table 8.1, or any other convenient root-finding technique to find a single root w_1 such that $w_1^3 = w^3$. Then the other two roots come easier. They are $e^{\pm i2\pi/3}w_1$; but $e^{\pm i2\pi/3} = (-1 \pm i\sqrt{3})/2$, so

$$w = w_1, \frac{-1 \pm i\sqrt{3}}{2} w_1. \tag{10.14}$$

We should observe, incidentally, that nothing prevents two actual roots of a cubic polynomial from having the same value. This certainly is possible, and it does not mean that one of the two roots is superfluous or that the polynomial has fewer than three roots. For example, the cubic polynomial $(z - 1)(z - 1)(z - 2) = z^3 - 4z^2 + 5z - 2$ has roots at 1, 1 and 2, with a single root at $z = 2$ and a double root—that is, two roots—at $z = 1$. When this happens, the method of Table 10.1 properly yields the single root once and the double root twice, just as it ought to do.

10.4 Edge cases

Section 10.3 excepts the edge cases $P = 0$ and $P^3 = -Q^2$. Mostly the book does not worry much about edge cases, but the effects of these cubic edge cases seem sufficiently nonobvious that the book might include here a few words about them, if for no other reason than to offer the reader a model of how to think about edge cases on his own. Table 10.1 gives the quadratic solution

$$w^3 \equiv Q \pm \sqrt{Q^2 + P^3},$$

[6] Numerically, it can matter. As a simple rule, because w appears in the denominator of x's definition, when the two w^3 differ in magnitude one might choose the larger.

in which § 10.3 generally finds it sufficient to consider either of the two signs. In the edge case $P = 0$,

$$w^3 = 2Q \text{ or } 0.$$

In the edge case $P^3 = -Q^2$,

$$w^3 = Q.$$

Both edge cases are interesting. In this section, we shall consider first the edge cases themselves, then their effect on the proof of § 10.3.

The edge case $P = 0$, like the general non-edge case, gives two distinct quadratic solutions w^3. One of the two however is $w^3 = Q - Q = 0$, which is awkward in light of Table 10.1's definition that $x \equiv w - P/w$. For this reason, in applying the table's method when $P = 0$, one chooses the other quadratic solution, $w^3 = Q + Q = 2Q$.

The edge case $P^3 = -Q^2$ gives only the one quadratic solution $w^3 = Q$; or more precisely, it gives two quadratic solutions which happen to have the same value. This is fine. One merely accepts that $w^3 = Q$, and does not worry about choosing one w^3 over the other.

The double edge case, or *corner case,* arises where the two edges meet—where $P = 0$ and $P^3 = -Q^2$, or equivalently where $P = 0$ and $Q = 0$. At the corner, the trouble is that $w^3 = 0$ and that no alternate w^3 is available. However, according to (10.12), $x^3 = 2Q - 3Px$, which in this case means that $x^3 = 0$ and thus that $x = 0$ absolutely, no other x being possible. This implies the triple root $z = -a_2/3$.

Section 10.3 has excluded the edge cases from its proof of the sufficiency of a single w^3. Let us now add the edge cases to the proof. In the edge case $P^3 = -Q^2$, both w^3 are the same, so the one w^3 suffices by default because the other w^3 brings nothing different. The edge case $P = 0$ however does give two distinct w^3, one of which is $w^3 = 0$, which puts an awkward $0/0$ in the table's definition of x. We address this edge in the spirit of l'Hôpital's rule, by sidestepping it, changing P infinitesimally from $P = 0$ to $P = \epsilon$. Then, choosing the $-$ sign in the definition of w^3,

$$\begin{aligned}
w^3 &= Q - \sqrt{Q^2 + \epsilon^3} = Q - (Q)\left(1 + \frac{\epsilon^3}{2Q^2}\right) = -\frac{\epsilon^3}{2Q}, \\
w &= -\frac{\epsilon}{(2Q)^{1/3}}, \\
x &= w - \frac{\epsilon}{w} = -\frac{\epsilon}{(2Q)^{1/3}} + (2Q)^{1/3} = (2Q)^{1/3}.
\end{aligned}$$

But choosing the + sign,

$$\begin{aligned}
w^3 &= Q + \sqrt{Q^2 + \epsilon^3} = 2Q, \\
w &= (2Q)^{1/3}, \\
x &= w - \frac{\epsilon}{w} = (2Q)^{1/3} - \frac{\epsilon}{(2Q)^{1/3}} = (2Q)^{1/3}.
\end{aligned}$$

Evidently the roots come out the same, either way. This completes the proof.

10.5 Quartics

Having successfully extracted the roots of the general cubic polynomial, we now turn our attention to the general quartic. The kernel of the cubic technique lay in reducing the cubic to a quadratic. The kernel of the quartic technique lies likewise in reducing the quartic to a cubic. The details differ, though; and, strangely enough, in some ways the quartic reduction is actually the simpler.[7]

As with the cubic, one begins solving the quartic by changing the variable

$$x + h \leftarrow z \tag{10.15}$$

to obtain the equation

$$x^4 = sx^2 + px + q, \tag{10.16}$$

where

$$\begin{aligned}
h &\equiv -\frac{a_3}{4}, \\
s &\equiv -a_2 + 6\left(\frac{a_3}{4}\right)^2, \\
p &\equiv -a_1 + 2a_2\left(\frac{a_3}{4}\right) - 8\left(\frac{a_3}{4}\right)^3, \\
q &\equiv -a_0 + a_1\left(\frac{a_3}{4}\right) - a_2\left(\frac{a_3}{4}\right)^2 + 3\left(\frac{a_3}{4}\right)^4.
\end{aligned} \tag{10.17}$$

[7]Even stranger, historically Ferrari discovered it earlier [28, "Quartic equation"]. Apparently Ferrari discovered the quartic's resolvent cubic (10.22), which he could not solve until Tartaglia applied Vieta's transform to it. What motivated Ferrari to chase the quartic solution while the cubic solution remained still unknown, this writer does not know, but one supposes that it might make an interesting story.

The reason the quartic is simpler to reduce is probably related to the fact that $(1)^{1/4} = \pm 1, \pm i$, whereas $(1)^{1/3} = 1, (-1 \pm i\sqrt{3})/2$. The $(1)^{1/4}$ brings a much neater result, the roots lying nicely along the Argand axes. This may also be why the quintic is intractable—but here we trespass the professional mathematician's territory and stray from the scope of this book. See Ch. 6's footnote 6.

To reduce (10.16) further, one must be cleverer. Ferrari [28, "Quartic equation"] supplies the cleverness. The clever idea is to transfer some but not all of the sx^2 term to the equation's left side by

$$x^4 + 2ux^2 = (2u + s)x^2 + px + q,$$

where u remains to be chosen; then to complete the square on the equation's left side as in § 2.2, but with respect to x^2 rather than x, as

$$\left(x^2 + u\right)^2 = k^2x^2 + px + j^2, \tag{10.18}$$

where

$$\begin{aligned} k^2 &\equiv 2u + s, \\ j^2 &\equiv u^2 + q. \end{aligned} \tag{10.19}$$

Now, one must regard (10.18) and (10.19) properly. In these equations, s, p and q have definite values fixed by (10.17), but not so u, j or k. The variable u is completely free; we have introduced it ourselves and can assign it any value we like. And though j^2 and k^2 depend on u, still, even after specifying u we remain free at least to choose signs for j and k. As for u, though no choice would truly be wrong, one supposes that a wise choice might at least render (10.18) easier to simplify.

So, what choice for u would be wise? Well, look at (10.18). The left side of that equation is a perfect square. The right side would be, too, if it were that $p = \pm 2jk$; so, arbitrarily choosing the $+$ sign, we propose the constraint that

$$p = 2jk, \tag{10.20}$$

or, better expressed,

$$j = \frac{p}{2k}. \tag{10.21}$$

Squaring (10.20) and substituting for j^2 and k^2 from (10.19), we have that

$$p^2 = 4(2u + s)(u^2 + q);$$

or, after distributing factors, rearranging terms and scaling, that

$$0 = u^3 + \frac{s}{2}u^2 + qu + \frac{4sq - p^2}{8}. \tag{10.22}$$

Equation (10.22) is the *resolvent cubic,* which we know by Table 10.1 how to solve for u, and which we now specify as a second constraint. If the constraints (10.21) and (10.22) are both honored, then we can safely substitute (10.20) into (10.18) to reach the form

$$\left(x^2+u\right)^2 = k^2x^2 + 2jkx + j^2,$$

which is

$$\left(x^2+u\right)^2 = \left(kx+j\right)^2. \tag{10.23}$$

The resolvent cubic (10.22) of course yields three u not one, but the resolvent cubic is a voluntary constraint, so we can just pick one u and ignore the other two. Equation (10.19) then gives k (again, we can just pick one of the two signs), and (10.21) then gives j. With u, j and k established, (10.23) implies the quadratic

$$x^2 = \pm(kx+j) - u, \tag{10.24}$$

which (2.2) solves as

$$x = \pm\frac{k}{2} \pm_o \sqrt{\left(\frac{k}{2}\right)^2 \pm j - u}, \tag{10.25}$$

wherein the two $\pm$ signs are tied together but the third, $\pm_o$ sign is independent of the two. Equation (10.25), with the other equations and definitions of this section, reveals the four roots of the general quartic polynomial.

In view of (10.25), the change of variables

$$\begin{aligned} K &\leftarrow \frac{k}{2}, \\ J &\leftarrow j, \end{aligned} \tag{10.26}$$

improves the notation. Using the improved notation, Table 10.2 summarizes the complete quartic polynomial root extraction method.

10.6 Guessing the roots

It is entertaining to put pencil to paper and use Table 10.1's method to extract the roots of the cubic polynomial

$$0 = [z-1][z-i][z+i] = z^3 - z^2 + z - 1.$$

One finds that

$$\begin{aligned} z &= w + \frac{1}{3} - \frac{2}{3^2 w}, \\ w^3 &\equiv \frac{2\left(5+\sqrt{3^3}\right)}{3^3}, \end{aligned}$$

Table 10.2: The method to extract the four roots of the general quartic polynomial. (In the table, the resolvent cubic is solved for u by the method of Table 10.1, where any one of the three resulting u serves. Either of the two K similarly serves. Of the three $\pm$ signs in x's definition, the $\pm_o$ is independent but the other two are tied together, the four resulting combinations giving the four roots of the general quartic.)

$$
\begin{aligned}
0 &= z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \\
s &\equiv -a_2 + 6\left(\frac{a_3}{4}\right)^2 \\
p &\equiv -a_1 + 2a_2\left(\frac{a_3}{4}\right) - 8\left(\frac{a_3}{4}\right)^3 \\
q &\equiv -a_0 + a_1\left(\frac{a_3}{4}\right) - a_2\left(\frac{a_3}{4}\right)^2 + 3\left(\frac{a_3}{4}\right)^4 \\
0 &= u^3 + \frac{s}{2}u^2 + qu + \frac{4sq - p^2}{8} \\
K &\equiv \pm\frac{\sqrt{2u+s}}{2} \\
J &\equiv \frac{p}{4K} \\
x &\equiv \pm K \pm_o \sqrt{K^2 \pm J - u} \\
z &= x - \frac{a_3}{4}
\end{aligned}
$$

which says indeed that $z = 1, \pm i$, but just you try to simplify it! A more baroque, more impenetrable way to write the number 1 is not easy to conceive. One has found the number 1 but cannot recognize it. Figuring the square and cube roots in the expression numerically, the root of the polynomial comes mysteriously to 1.0000, but why? The root's symbolic form gives little clue.

In general no better way is known;[8] we are stuck with the cubic baroquity. However, to the extent that a cubic, a quartic, a quintic or any other polynomial has real, rational roots, a trick is known to sidestep Tables 10.1 and 10.2 and guess the roots directly. Consider for example the quintic polynomial

$$z^5 - \frac{7}{2}z^4 + 4z^3 + \frac{1}{2}z^2 - 5z + 3.$$

Doubling to make the coefficients all integers produces the polynomial

$$2z^5 - 7z^4 + 8z^3 + 1z^2 - \mathrm{0xA}z + 6,$$

which naturally has the same roots. If the roots are complex or irrational, they are hard to guess; but if any of the roots happens to be real and rational, it must belong to the set

$$\left\{ \pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2}, \pm\frac{6}{2} \right\}.$$

No other real, rational root is possible. Trying the several candidates on the polynomial, one finds that 1, -1 and $3/2$ are indeed roots. Dividing these out leaves a quadratic which is easy to solve for the remaining roots.

The real, rational candidates are the factors of the polynomial's trailing coefficient (in the example, 6, whose factors are ± 1, ± 2, ± 3 and ± 6) divided by the factors of the polynomial's leading coefficient (in the example, 2, whose factors are ± 1 and ± 2). The reason no other real, rational root is possible is seen[9] by writing $z = p/q$—where p and q are integers and the fraction p/q is fully reduced—then multiplying the nth-order polynomial by q^n to reach the form

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0,$$

[8]At least, no better way is known to this writer. If any reader can straightforwardly simplify the expression without solving a cubic polynomial of some kind, the author would like to hear of it.

[9]The presentation here is quite informal. We do not want to spend many pages on this.

where all the coefficients a_k are integers. Moving the q^n term to the equation's right side, we have that

$$\left(a_n p^{n-1} + a_{n-1} p^{n-2} q + \cdots + a_1 q^{n-1}\right) p = -a_0 q^n,$$

which implies that $a_0 q$ is a multiple of p. But by demanding that the fraction p/q be fully reduced, we have defined p and q to be *relatively prime* to one another—that is, we have defined them to have no factors but ± 1 in common—so, not only $a_0 q$ but a_0 itself is a multiple of p. By similar reasoning, a_n is a multiple of q. But if a_0 is a multiple of p, and a_n, a multiple of q, then p and q are factors of a_0 and a_n respectively. We conclude for this reason, as was to be demonstrated, that no real, rational root is possible except a factor of a_0 divided by a factor of a_n.

Such root-guessing is little more than an algebraic trick, of course, but it can be a pretty useful trick if it saves us the embarrassment of inadvertently expressing simple rational numbers in ridiculous ways.

(Reference: [26, § 3.2].)

One could write much more about higher-order algebra, but now that the reader has tasted the topic he may feel inclined to agree that, though the general methods this chapter has presented to solve cubics and quartics are interesting, further effort were nevertheless probably better spent elsewhere. The next several chapters turn to the topic of the matrix, harder but much more profitable, toward which we mean to put substantial effort.

Chapter 11

The matrix (to be written)

"Chapters 2 through 9 have established the foundations of applied mathematics, upon which from this chapter forward we begin to build." Such words begin the draft of Chapter 11, not yet included in the book.

Starting here, the book's next topic will be the matrix, for which four chapters are planned:

- The matrix (this chapter);
- Matrix rank and the Gauss-Jordan factorization;
- Inversion and eigenvalue;
- Matrix algebra.

In earlier drafts, the four matrix chapters started as two; they may end as more than four, and of course the chapters' titles and contents may, probably will, change. Plans after the matrix chapters are less firm:

- Vector algebra;
- The divergence theorem and Stokes' theorem;
- Vector calculus;
- Probability (including the Gaussian, Rayleigh and Poisson distributions and the basic concept of sample variance);
- the Gamma function;
- The Fourier transform (including the Gaussian pulse and the Laplace transform);

- Feedback control;
- The spatial Fourier transform;
- Transformations to speed series convergence (such as the Poisson sum formula, Mosig's summation-by-parts technique and the Watson transformation);
- The scalar wave (or Helmholtz) equation[1] $(\nabla^2 + k^2)\psi(\mathbf{r}) = f(\mathbf{r})$ and Green's functions;
- Weyl and Sommerfeld forms (that is, spatial Fourier transforms in which only two or even only one of the three dimensions is transformed) and the parabolic wave equation;
- Bessel (especially Hankel) functions.

The goal is to complete approximately the above list, plus appropriate additional topics as the writing brings them to light. Beyond the goal, a few more topics are conceivable:

- Stochastics;
- Numerical methods;
- The mathematics of energy conservation;
- Statistical mechanics (however, the treatment of ideal-gas particle speeds probably goes in the earlier Probability chapter);
- Rotational dynamics;
- Kepler's laws.

Yet further developments, if any, are hard to foresee.[2]

[1]As you may know, this is called the "scalar wave equation" because $\psi(\mathbf{r})$ and $f(\mathbf{r})$ are scalar functions, even though the independent variable $\mathbf{r}$ they are functions of is a vector. Vector wave equations also exist and are extremely interesting, for example in electromagnetics. However, those are typically solved by (cleverly) transforming them into superpositions of scalar wave equations. The scalar wave equation merits a chapter for this reason among others.

[2]Any plans—I should say, any wishes—beyond the topics listed are no better than daydreams. However, for my own notes if for no other reason, plausible topics include the following: the Laurent series; Hamiltonian mechanics; electromagnetics; the statics of materials; the mechanics of materials; fluid mechanics; advanced special functions;

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If you find a part of the book insufficiently rigorous, then that is another matter. I do not discourage such criticism and would be glad to hear it, but this book may not be well placed to meet it (the book might compromise by including a footnote which briefly suggests the outline of a more rigorous proof, but it tries not to distract the narrative by formalities which do not serve applications). If you want to detail Hölder spaces and Galois theory, or whatever, then my response is likely to be that there is already a surfeit of fine professional mathematics books in print; this just isn't that kind of book. On the other hand, the book does intend to derive every one of its results adequately from an applied perspective; if it fails to do so in your view, then maybe you and I should discuss the matter. Finding the right balance is not always easy.

At the time of this writing, readers are downloading the book at the rate of about eight thousand copies per year directly through `derivations.org`. Some fraction of those, plus others who have installed the book as a Debian package or have acquired the book through secondary channels, actually have read it; now you stand among them.

Write as appropriate. More to come.

THB

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thermodynamics and the mathematics of entropy; quantum mechanics; electric circuits; statistics. Life should be so long, eh? Well, we shall see. Like most authors perhaps, I write in my spare time, the supply of which is necessarily limited and unpredictable. (Family responsibilities and other duties take precedence. My wife says to me, "You have a lot of chapters to write. It will take you a long time." She understates the problem.) The book targets the list ending in Bessel functions as its actual goal.

Appendix A

Hexadecimal and other notational matters

The importance of conventional mathematical notation is hard to overstate. Such notation serves two distinct purposes: it conveys mathematical ideas from writer to reader; and it concisely summarizes complex ideas on paper to the writer himself. Without the notation, one would find it difficult even to think clearly about the math; to discuss it with others, nearly impossible.

The right notation is not always found at hand, of course. New mathematical ideas occasionally find no adequate preëstablished notation, when it falls to the discoverer and his colleagues to establish new notation to meet the need. A more difficult problem arises when old notation exists but is inelegant in modern use.

Convention is a hard hill to climb. Rightly so. Nevertheless, slavish devotion to convention does not serve the literature well; for how else can notation improve over time, if writers will not incrementally improve it? Consider the notation of the algebraist Girolamo Cardano in his 1539 letter to Tartaglia:

> ... [T]he cube of one-third of the coefficient of the unknown is greater in value than the square of one-half of the number... [2]

If Cardano lived today, surely he would express the same thought in the form

$$\left(\frac{a}{3}\right)^3 > \left(\frac{x}{2}\right)^2.$$

Good notation matters.

Although this book has no brief to overhaul applied mathematical notation generally, it does seek to aid the honorable cause of notational evolution in a few specifics. For example, the book sometimes treats 2π implicitly as a single symbol, so that (for instance) the quarter revolution or right angle is expressed as $2\pi/4$ rather than as the less evocative $\pi/2$.

As a single symbol, of course, 2π remains a bit awkward. One wants to introduce some new symbol $\xi = 2\pi$ thereto. However, it is neither necessary nor practical nor desirable to leap straight to notational Utopia in one great bound. It suffices in print to improve the notation incrementally. If this book treats 2π sometimes as a single symbol—if such treatment meets the approval of slowly evolving convention—then further steps, the introduction of new symbols ξ and such, can safely be left incrementally to future writers.

A.1 Hexadecimal numerals

Treating 2π as a single symbol is a small step, unlikely to trouble readers much. A bolder step is to adopt from the computer science literature the important notational improvement of the hexadecimal numeral. No incremental step is possible here; either we leap the ditch or we remain on the wrong side. In this book, we choose to leap.

Traditional decimal notation is unobjectionable for measured quantities like 63.7 miles, \$ 1.32 million or $9.81\,\mathrm{m/s^2}$, but its iterative tenfold structure meets little or no aesthetic support in mathematical theory. Consider for instance the decimal numeral 127, whose number suggests a significant idea to the computer scientist, but whose decimal notation does nothing to convey the notion of the largest signed integer storable in a byte. Much better is the base-sixteen hexadecimal notation 0x7F, which clearly expresses the idea of $2^7 - 1$. To the reader who is not a computer scientist, the aesthetic advantage may not seem immediately clear from the one example, but consider the decimal number 2,147,483,647, which is the largest signed integer storable in a standard thirty-two bit word. In hexadecimal notation, this is 0x7FFF FFFF, or in other words $2^{\mathrm{0x1F}} - 1$. The question is: which notation more clearly captures the idea?

To readers unfamiliar with the hexadecimal notation, to explain very briefly: hexadecimal represents numbers not in tens but rather in sixteens. The rightmost place in a hexadecimal numeral represents ones; the next place leftward, sixteens; the next place leftward, sixteens squared; the next, sixteens cubed, and so on. For instance, the hexadecimal numeral 0x1357 means “seven, plus five times sixteen, plus thrice sixteen times sixteen, plus once sixteen times sixteen times sixteen.” In hexadecimal, the sixteen symbols 0123456789ABCDEF respectively represent the numbers zero through fifteen, with sixteen being written 0x10.

All this raises the sensible question: why sixteen?[1] The answer is that sixteen is 2^4, so hexadecimal (base sixteen) is found to offer a convenient shorthand for binary (base two, the fundamental, smallest possible base). Each of the sixteen hexadecimal digits represents a unique sequence of exactly four bits (binary digits). Binary is inherently theoretically interesting, but direct binary notation is unwieldy (the hexadecimal number 0x1357 is binary 0001 0011 0101 0111), so hexadecimal is written in proxy.

The conventional hexadecimal notation is admittedly a bit bulky and unfortunately overloads the letters A through F, letters which when set in italics usually represent coefficients not digits. However, the real problem with the hexadecimal notation is not in the notation itself but rather in the unfamiliarity with it. The reason it is unfamiliar is that it is not often encountered outside the computer science literature, but it is not encountered because it is not used, and it is not used because it is not familiar, and so on in a cycle. It seems to this writer, on aesthetic grounds, that this particular cycle is worth breaking, so this book uses the hexadecimal for integers larger than 9. If you have never yet used the hexadecimal system, it is worth your while to learn it. For the sake of elegance, at the risk of challenging entrenched convention, this book employs hexadecimal throughout.

Observe that in some cases, such as where hexadecimal numbers are arrayed in matrices, this book may omit the cumbersome hexadecimal prefix "0x." Specific numbers with physical dimensions attached appear seldom in this book, but where they do naturally decimal not hexadecimal is used: $v_{\mathrm{sound}} = 331$ m/s rather than the silly looking $v_{\mathrm{sound}} = \mathrm{0x14B}$ m/s.

Combining the hexadecimal and 2π ideas, we note here for interest's sake that

$$2\pi \approx \mathrm{0x6.487F}.$$

A.2 Avoiding notational clutter

Good applied mathematical notation is not cluttered. Good notation does not necessarily include every possible limit, qualification, superscript and

[1] An alternative advocated by some eighteenth-century writers was twelve. In base twelve, one quarter, one third and one half are respectively written 0.3, 0.4 and 0.6. Also, the hour angles (§ 3.6) come in neat increments of $(0.06)(2\pi)$ in base twelve, so there are some real advantages to that base. Hexadecimal, however, besides having momentum from the computer science literature, is preferred for its straightforward proxy of binary.

subscript. For example, the sum

$$S = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij}^2$$

might be written less thoroughly but more readably as

$$S = \sum_{i,j} a_{ij}^2$$

if the meaning of the latter were clear from the context.

When to omit subscripts and such is naturally a matter of style and subjective judgment, but in practice such judgment is often not hard to render. The balance is between showing few enough symbols that the interesting parts of an equation are not obscured visually in a tangle and a haze of redundant little letters, strokes and squiggles, on the one hand; and on the other hand showing enough detail that the reader who opens the book directly to the page has a fair chance to understand what is written there without studying the whole book carefully up to that point. Where appropriate, this book often condenses notation and omits redundant symbols.

Appendix B

The Greek alphabet

Mathematical experience finds the Roman alphabet to lack sufficient symbols to write higher mathematics clearly. Although not completely solving the problem, the addition of the Greek alphabet helps. See Table B.1.

When first seen in mathematical writing, the Greek letters take on a wise, mysterious aura. Well, the aura is fine—the Greek letters are pretty—but don't let the Greek letters throw you. They're just letters. We use

Table B.1: The Roman and Greek alphabets.

ROMAN

Aa	Aa	Gg	Gg	Mm	Mm	Tt	Tt
Bb	Bb	Hh	Hh	Nn	Nn	Uu	Uu
Cc	Cc	Ii	Ii	Oo	Oo	Vv	Vv
Dd	Dd	Jj	Jj	Pp	Pp	Ww	Ww
Ee	Ee	Kk	Kk	Qq	Qq	Xx	Xx
Ff	Ff	$L\ell$	Ll	Rr	Rr	Yy	Yy
				Ss	Ss	Zz	Zz

GREEK

$\mathrm{A}\alpha$	alpha	$\mathrm{H}\eta$	eta	$\mathrm{N}\nu$	nu	$\mathrm{T}\tau$	tau
$\mathrm{B}\beta$	beta	$\Theta\theta$	theta	$\Xi\xi$	xi	$\Upsilon\upsilon$	upsilon
$\Gamma\gamma$	gamma	$\mathrm{I}\iota$	iota	$\mathrm{O}o$	omicron	$\Phi\phi$	phi
$\Delta\delta$	delta	$\mathrm{K}\kappa$	kappa	$\Pi\pi$	pi	$\mathrm{X}\chi$	chi
$\mathrm{E}\epsilon$	epsilon	$\Lambda\lambda$	lambda	$\mathrm{P}\rho$	rho	$\Psi\psi$	psi
$\mathrm{Z}\zeta$	zeta	$\mathrm{M}\mu$	mu	$\Sigma\sigma$	sigma	$\Omega\omega$	omega

them not because we want to be wise and mysterious[1] but rather because

[1]Well, you can use them to be wise and mysterious if you want to. It's kind of fun, actually, when you're dealing with someone who doesn't understand math—if what you want is for him to go away and leave you alone. Otherwise, we tend to use Roman and Greek letters in various conventional ways: lower-case Greek for angles; capital Roman for matrices; e for the natural logarithmic base; f and g for unspecified functions; i, j, k, m, n, M and N for integers; P and Q for metasyntactic elements (the mathematical equivalents of `foo` and `bar`); t, T and τ for time; d, δ and Δ for change; A, B and C for unknown coefficients; J and Y for Bessel functions; etc. Even with the Greek, there are still not enough letters, so each letter serves multiple conventional roles: for example, i as an integer, an a-c electric current, or—most commonly—the imaginary unit, depending on the context. Cases even arise where a quantity falls back to an alternate traditional letter because its primary traditional letter is already in use: for example, the imaginary unit falls back from i to j where the former represents an a-c electric current.

This is not to say that any letter goes. If someone wrote

$$e^2 + \pi^2 = O^2$$

for some reason instead of the traditional

$$a^2 + b^2 = c^2$$

for the Pythagorean theorem, you would not find that person's version so easy to read, would you? Mathematically, maybe it doesn't matter, but the choice of letters is not a we simply do not have enough Roman letters. An equation like

$$\alpha^2 + \beta^2 = \gamma^2$$

says no more than does an equation like

$$a^2 + b^2 = c^2,$$

after all. The letters are just different.

Incidentally, we usually avoid letters like the Greek capital H (eta), which looks just like the Roman capital H, even though H (eta) is an entirely proper member of the Greek alphabet. Mathematical symbols are useful to us only inasmuch as we can visually tell them apart.

matter of arbitrary utility only but also of convention, tradition and style: one of the early writers in a field has chosen some letter—who knows why?—then the rest of us follow. This is how it usually works.

When writing notes for your own personal use, of course, you can use whichever letter you want. Probably you will find yourself using the letter you are used to seeing in print, but a letter is a letter; any letter will serve. When unsure which letter to use, just pick one; you can always go back and change the letter in your notes later if you want.

Appendix C

Manuscript history

The book in its present form is based on various unpublished drafts and notes of mine, plus some of my wife Kristie's (née Hancock), going back to 1983 when I was fifteen years of age. What prompted the contest I can no longer remember, but the notes began one day when I challenged a high-school classmate to prove the quadratic formula. The classmate responded that he didn't need to prove the quadratic formula because the proof was in the class math textbook, then counterchallenged me to prove the Pythagorean theorem. Admittedly obnoxious (I was fifteen, after all) but not to be outdone, I whipped out a pencil and paper on the spot and started working. But I found that I could not prove the theorem that day.

The next day I did find a proof in the school library,[1] writing it down, adding to it the proof of the quadratic formula plus a rather inefficient proof of my own invention to the law of cosines. Soon thereafter the school's chemistry instructor happened to mention that the angle between the tetrahedrally arranged four carbon-hydrogen bonds in a methane molecule was 109°, so from a symmetry argument I proved that result to myself, too, adding it to my little collection of proofs. That is how it started.[2]

The book actually has earlier roots than these. In 1979, when I was twelve years old, my father bought our family's first eight-bit computer. The computer's built-in *BASIC* programming-language interpreter exposed

[1] A better proof is found in § 2.10.

[2] Fellow gear-heads who lived through that era at about the same age might want to date me against the disappearance of the slide rule. Answer: in my country, or at least at my high school, I was three years too young to use a slide rule. The kids born in 1964 learned the slide rule; those born in 1965 did not. I wasn't born till 1967, so for better or for worse I always had a pocket calculator in high school. My family had an eight-bit computer at home, too, as we shall see.

functions for calculating sines and cosines of angles. The interpreter's manual included a diagram much like Fig. 3.1 showing what sines and cosines were, but it never explained how the computer went about calculating such quantities. This bothered me at the time. Many hours with a pencil I spent trying to figure it out, yet the computer's trigonometric functions remained mysterious to me. When later in high school I learned of the use of the Taylor series to calculate trigonometrics, into my growing collection of proofs the series went.

Five years after the Pythagorean incident I was serving the U.S. Army as an enlisted troop in the former West Germany. Although those were the last days of the Cold War, there was no shooting war at the time, so the duty was peacetime duty. My duty was in military signal intelligence, frequently in the middle of the German night when there often wasn't much to do. The platoon sergeant wisely condoned neither novels nor cards on duty, but he did let the troops read the newspaper after midnight when things were quiet enough. Sometimes I used the time to study my German—the platoon sergeant allowed this, too—but I owned a copy of Richard P. Feynman's *Lectures on Physics* [10] which I would sometimes read instead.

Late one night the battalion commander, a lieutenant colonel and West Point graduate, inspected my platoon's duty post by surprise. A lieutenant colonel was a highly uncommon apparition at that hour of a quiet night, so when that old man appeared suddenly with the sergeant major, the company commander and the first sergeant in tow—the last two just routed from their sleep, perhaps—surprise indeed it was. The colonel may possibly have caught some of my unlucky fellows playing cards that night—I am not sure—but me, he caught with my boots unpolished, reading the *Lectures.*

I snapped to attention. The colonel took a long look at my boots without saying anything, as stormclouds gathered on the first sergeant's brow at his left shoulder, then asked me what I had been reading.

"Feynman's *Lectures on Physics,* sir."

"Why?"

"I am going to attend the university when my three-year enlistment is up, sir."

"I see." Maybe the old man was thinking that I would do better as a scientist than as a soldier? Maybe he was remembering when he had had to read some of the *Lectures* himself at West Point. Or maybe it was just the singularity of the sight in the man's eyes, as though he were a medieval knight at bivouac who had caught one of the peasant levies, thought to be illiterate, reading Cicero in the original Latin. The truth of this, we shall never know. What the old man actually said was, "Good work, son. Keep it up."

The stormclouds dissipated from the first sergeant's face. No one ever said anything to me about my boots (in fact as far as I remember, the first sergeant—who saw me seldom in any case—never spoke to me again). The platoon sergeant thereafter explicitly permitted me to read the *Lectures* on duty after midnight on nights when there was nothing else to do, so in the last several months of my military service I did read a number of them. It is fair to say that I also kept my boots better polished.

In Volume I, Chapter 6, of the *Lectures* there is a lovely introduction to probability theory. It discusses the classic problem of the "random walk" in some detail, then states without proof that the generalization of the random walk leads to the Gaussian distribution

$$p(x) = \frac{\exp(-x^2/2\sigma^2)}{\sigma\sqrt{2\pi}}.$$

For the derivation of this remarkable theorem, I scanned the book in vain. One had no Internet access in those days, but besides a well equipped gym the Army post also had a tiny library, and in one yellowed volume in the library—who knows how such a book got there?—I did find a derivation of the $1/\sigma\sqrt{2\pi}$ factor.[3] The exponential factor, the volume did not derive. Several days later, I chanced to find myself in Munich with an hour or two to spare, which I spent in the university library seeking the missing part of the proof, but lack of time and unfamiliarity with such a German site defeated me. Back at the Army post, I had to sweat the proof out on my own over the ensuing weeks. Nevertheless, eventually I did obtain a proof which made sense to me. Writing the proof down carefully, I pulled the old high-school math notes out of my military footlocker (for some reason I had kept the notes and even brought them to Germany), dusted them off, and added to them the new Gaussian proof.

That is how it has gone. To the old notes, I have added new proofs from time to time; and although somehow I have misplaced the original high-school leaves I took to Germany with me, the notes have nevertheless grown with the passing years. After I had left the Army, married, taken my degree at the university, begun work as a building construction engineer, and started a family—when the latest proof to join my notes was a mathematical justification of the standard industrial construction technique for measuring the resistance-to-ground of a new building's electrical grounding system—I was delighted to discover that Eric W. Weisstein had compiled

[3] The citation is now unfortunately long lost.

and published [28] a wide-ranging compilation of mathematical results in a spirit not entirely dissimilar to that of my own notes. A significant difference remained, however, between Weisstein's work and my own. The difference was and is fourfold:

1. Number theory, mathematical recreations and odd mathematical names interest Weisstein much more than they interest me; my own tastes run toward math directly useful in known physical applications. The selection of topics in each body of work reflects this difference.

2. Weisstein often includes results without proof. This is fine, but for my own part I happen to like proofs.

3. Weisstein lists results encyclopedically, alphabetically by name. I organize results more traditionally by topic, leaving alphabetization to the book's index, that readers who wish to do so can coherently read the book from front to back.[4]

4. I have eventually developed an interest in the free-software movement, joining it as a Debian Developer [7]; and by these lights and by the standard of the Debian Free Software Guidelines (DFSG) [8], Weisstein's work is not free. No objection to non-free work as such is raised here, but the book you are reading *is* free in the DFSG sense.

A different mathematical reference, even better in some ways than Weisstein's and (if I understand correctly) indeed free in the DFSG sense, is emerging at the time of this writing in the on-line pages of the general-purpose encyclopedia Wikipedia [3]. Although Wikipedia remains generally uncitable[5] and forms no coherent whole, it is laden with mathematical

[4]There is an ironic personal story in this. As children in the 1970s, my brother and I had a 1959 World Book encyclopedia in our bedroom, about twenty volumes. It was then a bit outdated (in fact the world had changed tremendously in the fifteen or twenty years following 1959, so the encyclopedia was more than a bit outdated) but the two of us still used it sometimes. Only years later did I learn that my father, who in 1959 was fourteen years old, had bought the encyclopedia with money he had earned delivering newspapers daily before dawn, *and then had read the entire encyclopedia, front to back.* My father played linebacker on the football team and worked a job after school, too, so where he found the time or the inclination to read an entire encyclopedia, I'll never know. Nonetheless, it does prove that even an encyclopedia can be read from front to back.

[5]Some "Wikipedians" do seem actively to be working on making Wikipedia authoritatively citable. The underlying philosophy and basic plan of Wikipedia admittedly tend to thwart their efforts, but their efforts nevertheless seem to continue to progress. We shall see. Wikipedia is a remarkable, monumental creation.

knowledge, including many proofs, which I have referred to more than a few times in the preparation of this text.

This book is bound to lose at least a few readers for its unorthodox use of hexadecimal notation ("The first primes are $2, 3, 5, 7, \mathrm{0xB}, \ldots$"). Perhaps it will gain a few readers for the same reason; time will tell. I started keeping my own theoretical math notes in hex a long time ago; at first to prove to myself that I could do hexadecimal arithmetic routinely and accurately with a pencil, later from aesthetic conviction that it was the right thing to do. Like other applied mathematicians, I've several own private notations, and in general these are not permitted to burden the published text. The hex notation is not my own, though. It existed before I arrived on the scene, and since I know of no math book better positioned to risk its use, I have with hesitation and no little trepidation resolved to let this book do it. Some readers will approve; some will tolerate; undoubtedly some will do neither. The views of the last group must be respected, but in the meantime the book has a mission; and crass popularity can be only one consideration, to be balanced against other factors. The book might gain even more readers, after all, had it no formulas, and painted landscapes in place of geometric diagrams! I like landscapes, too, but anyway you can see where that line of logic leads.

More substantively: despite the book's title, adverse criticism from some quarters for lack of rigor is probably inevitable; nor is such criticism necessarily improper from my point of view. Still, serious books by professional mathematicians tend to be *for* professional mathematicians, which is understandable but does not always help the scientist or engineer who wants to use the math to model something. The ideal author of such a book as this would probably hold two doctorates: one in mathematics and the other in engineering or the like. The ideal author lacking, I have written the book.

So here you have my old high-school notes, extended over twenty years and through the course of two-and-a-half university degrees, now partly typed and revised for the first time as a LaTeX manuscript. Where this manuscript will go in the future is hard to guess. Perhaps the revision you are reading is the last. Who can say? The manuscript met an uncommonly enthusiastic reception at Debconf 6 [7] May 2006 at Oaxtepec, Mexico; and in August of the same year it warmly welcomed Karl Sarnow and Xplora Knoppix [29] aboard as the second official distributor of the book. Such developments augur well for the book's future at least. But in the meantime, if anyone should challenge you to prove the Pythagorean theorem on the spot, why, whip this book out and turn to § 2.10. That should confound

’em.

THB

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Index